EVOLVING CONVEX HYPERSURFACES

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A thesis submitted for the degree of Doctor of Philosophy
of The Australian National University.

February 1993

This thesis describes my own work except where otherwise indicated. Theorems (II.5–27) and (III.1–3) are partly due to Gerhard Huisken. Several sections use or modify techniques developed by others, as I have indicated in the text.

I wish to thank Gerhard Huisken for his advice and guidance over the last three years. Thanks are also due to Neil Trudinger, John Hutchinson, Klaus Ecker, John Urbas, Leon Simon, Robert Bartnik, Richard Hamilton, Steven Altschuler, Lang-Fang Wu, Giovanni Gregori and Peter Leviton, for useful discussions, advice, inspiration and encouragement. Finally, thanks to Kylie and the cats for their company and patience.

Abstract

The subject of this thesis is the deformation of hypersurfaces by means of geometrically defined parabolic equations. In the most general case, we consider hypersurfaces moving in a Riemannian manifold, with speed determined by a function of the Weingarten curvature. The majority of the thesis concerns the case of convex hypersurfaces.

Section I of the thesis concerns evolution equations which generalise, in a certain sense, the well-known mean curvature flow : We consider the motion of hypersurfaces in Euclidean space, where the speed is a function of the principal curvatures. As for the mean curvature flow, we require this function to be homogeneous of degree one, and strictly increasing in each argument. The motion is then described by a fully nonlinear parabolic equation. Under natural structure conditions on the equations and natural convexity conditions on the initial hypersurface, it is shown that a unique solution exists for a finite time; this solution converges uniformly to a point and becomes spherical in shape towards the final time. This result generalises work on the mean curvature flow by Gerhard Huisken, and related work on other particular flows by Ben Chow. The proof employed is in some respects similar to these earlier cases, but achieves important simplifications through the use of a new result concerning locally pinched convex hypersurfaces.

In section II, we consider a much wider class of parabolic evolution equations, allowing not only other degrees of homogeneity for the speed, but also nonhomogeneous equations, and speeds depending on the normal direction at each point as well as the Weingarten curvature. Precise Harnack estimates are proved for a very wide class of such equations, characterised by simple structure condi-

tions. In the proof, the Gauss map is used to parametrise the hypersurfaces. This change in parametrisation results in remarkable simplification and clarification of the calculations. In contrast, long and complicated calculations were required by Hamilton and Chow in their proofs of special cases of the Harnack inequalities. Some entropy inequalities are also proved here for special flows, and the calculation of the Harnack inequalities is extended to the case of complete convex hypersurfaces.

Section III gives results for a wide variety of flows: The first chapter deals with a class of contraction flows, showing under appropriate conditions that convex hypersurfaces contract to points. The next chapter concentrates on contracting curves (a case not considered in section I). A natural class of anisotropic flows is considered, allowing homogeneity of degree greater than or equal to one in the curvature. It is shown that embedded convex curves have the expected limiting behaviour under such equations. The results are proved using generalisations of methods due to Gage. The third chapter concerns anisotropic expansion flows, showing under appropriate conditions that star-shaped hypersurfaces expand to infinite radius, converging to the expected limiting shape as they do so. This generalises results for the isotropic case due to Gerhardt and Urbas.

Section IV uses the Gauss map techniques to give an elegant new proof of the Aleksandrov-Fenchel inequalities for mixed volumes of convex regions. The proof uses special evolution equations whose form is suggested by expressions for the mixed volumes. The proof is significantly simpler than those previously available.

In section V it is shown that there is an important connection between entropy inequalities and the Aleksandrov-Fenchel inequalities. New entropy inequalities are proved for many evolution equations, by a new proof which directly uses the Aleksandrov-Fenchel inequalities. These estimates are applied to expansion flows of curves in the plane, with speeds homogeneous of degree less than minus one in the curvature. It is shown that solutions expand to infinity in finite time, and that they approach spheres (or other limit shapes for anisotropic equations) after rescaling. Another interesting consequence is that solutions to contraction flows with small degree of homogeneity do not in general converge to the expected limiting shape near the final singularity.

The last section concerns hypersurfaces in Riemannian background spaces. The techniques of section I are adapted to this more difficult situation, giving good results for a somewhat restricted class of flows. A strictly convex, compact initial hypersurface, in a space with non-negative sectional curvatures, gives a solution which contracts to a point and becomes round. Also, slightly different flows are used to give the same result for an initial hypersurface with all principal curvatures greater than 1, in a background space with all sectional curvatures greater than or equal to -1. It follows that any such hypersurface is the boundary of an immersed disc. This gives an elegant new proof of the 1/4-pinching sphere theorem of Klingenberg, Berger and Rauch, and also proves a generalised "dented sphere" theorem which allows some negative curvature.

Contents

INTRODUCTION

In this thesis we consider hypersurfaces evolving under geometrically defined parabolic equations. The motion of a hypersurface is described by a family of immersions $\varphi : M^n \times [0,T) \to N^{n+1}$, where M^n is an n-dimensional manifold, and N^{n+1} is a Riemannian manifold of dimension $n+1$. The evolution equations to be considered depend only on the geometry of the immersions in N, and are invariant under diffeomorphisms of M. A well-known example of such an evolution equation is the mean curvature flow, which deforms a prescribed initial immersion φ_0 of M to give a family of immersions φ satisfying the following equations:

$$\frac{\partial}{\partial t}\varphi(x,t) = -H(x,t)\nu(x,t) \tag{1}$$
$$\varphi(x,0) = \varphi_0(x)$$

for every $(x,t) \in M \times [0,T)$, where $\nu(x,t)$ is a unit normal to the hypersurface $\varphi(M,t)$ at x, and $H(x,t)$ is the mean curvature of $\varphi(M,t)$ at x.

The mean curvature flow was the first flow of this kind to be considered, and still receives more attention than any other example. There are several reasons for this: It has a somewhat simpler definition and structure than most other examples, and is naturally motivated as the gradient flow of the area functional. This links the theory of solutions of the mean curvature flow with that of minimal surfaces. The variational definition makes sense for arbitrary hypersurfaces, and even for more general objects from geometric measure theory—such a general approach was adopted by Brakke [**Br**] who investigated many of the general features of solutions. There are also some applications to physical situations where area is important—the mean curvature flow has been proposed as a model for the evolution of grain boundaries in annealing metals; more recently it has been proved by Evans, Soner, and Souganidis [**ESS**] and Ilmanen [**Il**] that the mean curvature flow appears as a limiting case of an equation describing the motion of phase boundaries by isotropic surface tension. Another important reason for interest in the mean curvature flow has been the potential for applications in geometry. This idea is one of the main sources of motivation for the present thesis.

The prospects for geometric applications seem promising: Particularly encouraging is a beautiful result of Huisken [**Hu1**] which provides a detailed qualitative description of the behaviour of solutions to the mean curvature flow, in the special case of compact, convex hypersurfaces in Euclidean space. He showed that for any such initial hypersurface (of dimension $n \geq 2$), there exists a unique smooth solution to the mean curvature flow which converges to a point in finite time, in such a way that the hypersurfaces become spherical as the final time is approached. An analogous result for embedded convex curves in the plane (the case $n = 1$) was proved by Gage and Hamilton ([**Ga1–2**], [**GH**]), and extended by Grayson [**Gr**] to non-convex embedded curves. These results make clear the important regularising behaviour of the flow.

There is a great deal of research currently in progress with the aim of describing the behaviour of solutions to the mean curvature flow, in the more general case of non-convex hypersurfaces (in the case $n \geq 2$). It has become clear that such a description will involve considerably greater technical difficulty than the convex case, but nevertheless there is the possibility that very general and powerful results may be obtained.

In this thesis we pursue a different approach, and concentrate on applications which make use of convex hypersurfaces. This meets with considerable success—sections IV and VI use evolving hypersurfaces to prove important results in quite different areas of geometry. In achieving this it is found necessary to adopt a wider perspective, considering not only the mean curvature flow, but a much more general class of parabolic evolution equations.

There is another reason for considering such a general class of equations, which is itself another main source of motivation for the thesis: The equations considered are, in general, fully nonlinear parabolic equations (the mean curvature flow itself is quasi-linear). The elliptic counterparts of these equations have received much attention as interesting examples of fully nonlinear equations—from the Minkowski problem and its generalisations, to equations of prescribed curvature for star-shaped hypersurfaces and graphs. This thesis shows that many fully nonlinear parabolic equations are also of considerable interest. We make extensive use of the recent developments in the regularity theory for fully nonlinear equations, as described by Krylov [**K**].

The result of Huisken mentioned above has served as a model for many of the most successful analyses of convex hypersurface flows to date. Some other methods have achieved partial success, such as those due to Tso [**Ts**] who considered the flow by Gauss curvature, given by (1–1) with H replaced by the Gauss curvature K. This flow differs from the mean curvature flow in several important respects, such as in the different degree of homogeneity of the speed. Tso was able to establish that solutions converge to points in finite time under this flow, but the question of whether the hypersurfaces become spherical remains open. Chow adapted the results of Tso to flows by arbitrary positive powers of the Gauss curvature, with the same results [**Ch1**]. In the special case of the flow by the nth root of the Gauss curvature (which has the same homogeneity as the mean curvature), Chow was able to use the techniques developed by Huisken to prove that the hypersurfaces become spherical. He later used these techniques again for the flow by the square root of the scalar curvature [**Ch2**], proving convergence to a point and roundness in the limit, provided the principal curvatures of the initial hypersurface satisfy a certain pinching condition.

From these results it is clear that the degree of homogeneity of the speed is critical to the success of Huisken's techniques. This general situation—hypersurfaces in Euclidean space evolving with speeds homogeneous of degree one in the principal curvatures—is the subject of the first section of this thesis. The main result is that the qualitative behaviour of contraction to a point and roundness in the limit (for $n \geq 2$) is shared by all evolution equations satisfying some simple structure conditions (theorem (I.1–2)).

In the proof of this result, some of the techniques are similar to those in [**Hu1**] and [**Ch1–2**]. In particular, the parabolic maximum principle is the main tool in showing that convexity is preserved under the evolution and that a local pinching condition holds, although the details of the calculations are different in some cases. The proof overall is significantly simpler than in these earlier papers, however. The main simplification comes from a short lemma concerning compact, locally pinched, convex hypersurfaces. This elegant result makes the hardest sections of the earlier proofs redundant: It is no longer necessary to use the techniques of Stampacchia iteration and the Michael-Simon Sobolev inequality, or the complicated argument in which they were required—these are replaced by relatively simple and intuitive geometric considerations.

These flows of mean curvature type are by no means the only hypersurface flows of interest. Some other important examples of other flows are given as the gradient flows of various natural energies associated with convex hypersurfaces: For each integer k between 1 and n there is a natural energy quantity V_k, called the kth mean cross-sectional volume. These are defined by the following integrals:

$$V_k(\varphi(M)) = \int_M e_{n-k}(\lambda_1, \ldots, \lambda_n)\, d\mu \tag{2}$$

where e_ℓ is the ℓth elementary symmetric function of the principal curvatures $\lambda_1, \ldots, \lambda_n$, and $d\mu$ is the measure given by the metric on the hypersurface $\varphi(M)$. The case $k = n$ gives the area functional. The resulting gradient flows depend on the L^2 space we choose—there are a family of natural L^2 spaces, given by the measures $e_\ell d\mu$ for $\ell = 0, \ldots, n$. The gradient flow of V_k with respect to the measure $e_\ell d\mu$ is the flow with speed $\frac{e_{k+1}}{e_\ell}$, for $k \geq \ell$. This class of flows includes the Gauss curvature flow, which is the gradient flow of the mean width V_1 with respect to the measure $d\mu$ on the hypersurface.

The Gauss curvature flow also appears as a model for some physical processes: Firey [**Fi**] proposed it as a model for the changing shape of an object being worn down by collisions from all directions, such as a pebble on a beach.

Another class of hypersurface evolution equations, with somewhat different behaviour, has been considered by Urbas [**U1-2**] and Gerhardt [**Ge**]. These are so-called expansion flows, where a hypersurface evolves in an outward direction, with a speed homogeneous of degree -1 in the principal curvatures. It has been shown that under very general structure conditions, such evolution equations have solutions which last for infinite time, becoming spherical in shape for large times. This result holds not only for convex initial data, but also for more general star-shaped hypersurfaces.

Models for the evolution of crystals (see for example [**CHT**]) lead to further equations of interest. For example, we may wish to consider the gradient flow of an energy which depends on the normal direction. The simplest such example, a kind of anisotropic area functional, is given by the following integral:

$$\int_M \phi(\nu) d\mu \tag{3}$$

where ϕ is a function defined on the sphere S^n, and $\nu : M^n \to S^n$ is the Gauss or normal mapping. The gradient flow of such an energy has a speed which depends on the normal direction.

Section two incorporates all of these examples in a much larger class of evolution equations of hypersurfaces, allowing speeds which are homogeneous of any degree in the curvature, or even non-homogeneous in the curvature; furthermore the speed is allowed to depend on the normal direction and the entire Weingarten curvature, rather than just the principal curvatures. It is remarkable that useful results can still be obtained under such general conditions. The main result proved in this section is a precise parabolic Harnack inequality. This is one of many Harnack inequalities which have been proved recently for solutions to geometric evolution equations: The first of this type was due to Li and Yau [**LY**], for the heat equation; see also the recent work of Hamilton [**Ha4**] which extends this. Hamilton has also proved Harnack inequalities for the Ricci flow on surfaces [**Ha2**], the curve shortening flow, the mean curvature flow, several scalar evolution equations [**Ha3**], and the Ricci flow [**Ha5**]. Chow [**Ch3**] extended the proof for the mean curvature flow to flows by positive powers of the Gauss curvature. Unfortunately the work of Hamilton and Chow involves long and complicated calculations, which obscure the elegance of the final result. Here we prove that Harnack inequalities hold for evolution equations satisfying simple and natural structure conditions. Furthermore, the proof is dramatically simplified by a geometrically natural parametrisation of the evolving hypersurfaces, using the Gauss map. With this simple change in parametrisation, the calculations become transparent. The same calculations lead to integral estimates, known as entropy inequalities, for solutions to certain special evolution equations. It is interesting to note that the simplest results for all these calculations hold not for the mean curvature flow, but for flow by powers of the harmonic mean of the principal curvatures. The section concludes by showing that the Harnack inequality holds also for suitable non-compact hypersurfaces. This is accomplished using techniques due to Ecker and Huisken [**EH**], who used them to prove interior estimates for the mean curvature flow.

Section three proves some results about the behaviour of the more general flow equations considered in part two: For many evolution equations with speeds homogeneous of positive degree in the curvature, it is still true that strictly convex initial hypersurfaces contract to points in finite time. The techniques used here are partly due to Tso [**Ts**] who developed them for the Gauss curvature flow. I also give special attention to contracting curves. I prove a generalisation of an isoperimetric inequality due to Gage [**Ga1**], and use it to prove good results for a wide variety of natural contraction flows of curves. These include flows by powers of the curvature (of degree greater than or equal to one), under which convex curves become spherical after rescaling. Natural anisotropic versions of these flows are also considered: Associated with any closed, embedded convex curve γ which is symmetric about the origin, there is a class of natural flows. As in the isotropic case, I show that those flows with degree of homogeneity greater than or equal to one give convergence to the limit shape γ. The last chapter of this section concerns a class of anisotropic expansion flows, analogous to the isotropic case considered by Gerhardt [**Ge**] and Urbas [**U1–2**]. For flows where the speed is homogeneous of degree less than or equal to one in the radii of curvature, the solutions converge to the expected limit shape.

In section IV, some special evolution equations are applied to give a new proof of some fundamental inequalities for convex regions of Euclidean space—the Aleksandrov-Fenchel inequalities. These inequalities have been known since 1936, and several proofs are available, all of which require considerable effort. This new proof is remarkably simple, requiring only two short calculations in conjunction with the general results of part three. In this application it is necessary to make use of much of the generality of part three—the speed functions depend on the normal direction in a non-trivial way. After proving the Aleksandrov-Fenchel inequalities in their simplest form, we use carefully chosen evolution equations to give direct and simple proofs for many of the most interesting consequences of the inequalities, including the isoperimetric inequality.

The fifth section uses the Aleksandrov-Fenchel inequalities to interpret the entropy inequalities proved in section two, and proves new entropy inequalities for a much larger class of equations. These entropy estimates give decreasing integral quantities for rescaled solutions to the evolution equations. I give some applications of these new estimates: The first is to expanding curves with speeds homogeneous of degree less than minus one in the principal curvatures. These are more difficult than the expansion flows with lower degree, as the solutions reach infinite size in finite time. With the aid of the entropy estimates, I show that the solutions converge to the expected limiting shape after rescaling. The second application is to contraction flows of small degree: I use the entropy estimates to prove that solutions to such flows do not in general converge to the expected limit shape.

Section VI concerns convex hypersurfaces in Riemannian spaces. Huisken **[Hu2]** has considered the mean curvature flow in this setting, adapting many of the techniques for hypersurfaces in Euclidean space to this general case. He proved that a compact, strictly convex hypersurface contracts to a point and becomes spherical under the mean curvature flow, provided the convexity of the initial hypersurface is sufficient to overcome any geometric obstacles of the background space. Unfortunately, it was necessary in these results to assume a convexity condition depending on the gradient of the Riemann tensor of the background space, as well as the sectional curvatures. In this section of the thesis it is shown that better results can be obtained in some cases: We consider a class of flows, not including the mean curvature flow, but including the flow by harmonic mean curvature. A result is obtained which no longer depends on the gradient of the Riemann tensor. The resulting convexity condition depends only on the sectional curvatures of the background space, and is easily seen to be sharp. In particular, it is shown that any strictly convex hypersurface in a Riemannian manifold with non-negative sectional curvatures is the boundary of an immersed disc; similar results hold for hypersurfaces in spaces with sectional curvatures bounded below, if we require a stronger convexity condition: If the sectional curvatures are greater than or equal to -1, it is sufficient to assume that the principal curvatures are greater than one. The proof of the latter case employs non-homogeneous flows. The result is used to give a new proof of the 1/4-pinching sphere theorem of Klingenberg, Berger and Rauch, and also to prove a new 'dented sphere' theorem which allows some negative curvature.

The notation of the thesis will be explained in several sections: Most of the notation for hypersurfaces in Euclidean space will be included in section I. Section II gives the notation for the Gauss map machinery, and some special notation is given in section IV for the mixed volumes and special flows. The notation for hypersurfaces in Riemannian space is only slightly different; this is summarised in section VI. Equations are given numbers depending on the chapter and section of the thesis in which they occur: For example, the third equation in the second chapter of part II is numbered (II.2–3). The number of the section will be omitted except when referring to equations in different parts of the thesis.

Section I

CONTRACTING CONVEX HYPERSURFACES IN EUCLIDEAN SPACE

1. Introduction.

This section deals with convex hypersurfaces moving through Euclidean space under evolution equations which have speeds homogeneous of degree one in the principal curvatures. The results proved here are modelled on an important result of Huisken [**Hu1**] concerning the behaviour of convex solutions of the mean curvature flow. The main aim will be to simplify the techniques developed by Huisken, and apply them to evolution equations in as wide a class as possible. The results obtained include all the previous results for flows with this homogeneity, and also include a wide variety of other interesting evolution equations which have not been studied before, such as the flow by harmonic mean curvature. This flow can be used to prove several useful geometrical results (see sections IV and VI).

Let M^n be a smooth, compact manifold of dimension $n \geq 2$ without boundary. Suppose $\varphi_0 : M^n \to \mathbb{R}^{n+1}$ is a smooth immersion of M^n which is strictly convex. A stronger convexity condition may be required for some evolution equations (see chapter 3 for the exact requirements). We seek a smooth family of immersions $\varphi : M^n \times [0,T) \to \mathbb{R}^{n+1}$ satisfying an equation of the following form:

$$\begin{aligned} &\frac{\partial}{\partial t}\varphi(x,t) = -F\left(\mathcal{W}(x,t)\right)\nu(x,t) \\ &\varphi(x,0) = \varphi_0(x) \end{aligned} \tag{1–1}$$

for all x in M^n and t in $[0,T)$. In this equation $\nu(x,t)$ is a unit normal to the hypersurface $M_t := \varphi_t(M^n)$ at the point $\varphi_t(x)$, where φ_t denotes the immersion at time t. $\mathcal{W}(x,t)$ is the Weingarten map of TM defined with respect to the normal ν (see the definition (2–7)), and $F : S_+ \subset T^*M \otimes TM \to R$ is a smooth function defined on the set S_+ of all symmetric, positive transformations of TM^n. F is assumed to satisfy natural structure conditions which are given in chapter 3. These conditions are similar to those used by Caffarelli, Nirenberg and Spruck for elliptic equations involving functions of the Hessian [**CNS1**] or the Weingarten curvature ([**CNS2**], [**CNS3**]). The class considered includes many of the classical curvatures. The main theorem of this section is the following:

Theorem 1–2. *Suppose φ_0 and F satisfy the conditions* (3–1). *There exists a unique, smooth solution to equation* (1–1) *on a maximal time interval* $[0, T)$. *The maps φ_t converge uniformly to a constant p in $\mathbb{R}^{n+1}$ as t approaches T. The rescaled immersions given by $\tilde{\varphi}_\tau = (2(T-t))^{-\frac{1}{2}}(\varphi_t - p)$ converge to a smooth embedding φ_∞ with image equal to the unit sphere in $\mathbb{R}^{n+1}$, exponentially in C^∞ with respect to the natural rescaled time parameter $\tau = -\frac{1}{2}\ln\left(1-\frac{t}{T}\right)$.*

The proof is organised as follows: Chapter 2 introduces the required notation and preliminary results for the proof. The notation adopted is somewhat different from standard, emphasising the geometric content of the equations as much as possible. In most instances indicial notation is not employed, although some attempt is made to accomodate those who are accustomed to such notation through the use of explanatory remarks and examples. The chapter defines the metric, connection, and Weingarten curvature, and introduces concepts such as the support function of a convex hypersurface. Also important is a discussion of hypersurfaces which can be given as a graph over a sphere (star-shaped hypersurfaces), and the expressions for various geometric quantities in terms of the graphical height function. Some important properties of symmetric functions are also given.

The precise definition of the class of evolution equations is given in chapter 3. This chapter also covers various other aspects of the evolution equations: A proof of uniqueness and short-time existence of solutions is given, by describing the evolving hypersurfaces as graphs of functions on the sphere; the induced evolution equation for these scalar functions is deduced and shown to be strictly parabolic. The equations which govern the evolution of geometric objects (the metric, normal and curvature of the evolving hypersurfaces) are deduced. An important result of this is a favourable evolution equation for the Weingarten curvature, which follows from a generalised Simons's identity. It is here that the homogeneity of the speed function is vitally important.

Chapter 4 contains an important step in the proof: It is shown that the convexity of the immersions is preserved as long as the solution exists, and also that a pointwise pinching estimate holds throughout the evolution. This step is similar in many ways to the corresponding section of [**Hu1**]—the parabolic maximum principle is the main tool, applied in different ways depending on the detailed structure of the evolution equations considered. It is in this step that most of the structure conditions are required—once a pinching condition is known, only very few conditions are required to complete the proof.

In the work of Huisken [**Hu1**] on the mean curvature flow, and the later work of Chow on other specific flows [**Ch1–2**], it was the application of this local pinching estimate which presented the most difficulties. Huisken used the result to prove an even stronger pinching estimate, showing that the pinching must improve as the magnitude of the curvature becomes large. This involved several steps: First, Simons's identity was used to prove an estimate for certain integral quantities, similar to a Poincaré inequality. This was then used in conjunction with the Michael-Simon Sobolev inequality ([**MS**]) in a careful argument to yield the required pinching estimate by the technique of Stampacchia iteration. Chow adapted this method to the cases he considered.

The proof presented here avoids this complicated machinery altogether, by applying the local pinching estimate directly to control the geometry of the hypersurfaces: In chapter 5, a simple lemma is proved which gives very strong control over the shape of a compact, locally pinched convex hypersurface. The proof of the lemma itself is very short, requiring only a simple calculation using integration by parts. It should be noted, however, that the result could not have been achieved without the machinery of the Gauss map parametrisation of convex hypersurfaces. The importance of the support function and the resulting description of the geometry of convex hypersurfaces is a theme which will recur in many parts of this thesis, and is vital to most of the results of later sections.

In chapter 6, this control over the geometry is applied, using the general regularity theory for nonlinear parabolic equations developed by Krylov and others [**K**], to prove that the solution converges to a point.

The proof of convergence to a sphere is then quite straightforward, and is given in chapter 7. The main ingredients for this are the results of the previous two chapters, with further application of general results from [**K**], and an estimate adapted from the work of Tso in [**Ts**].

2. Notation and Preliminary Results

Some general notation for manifolds and their associated tensor bundles will be required. For a manifold M, the tangent bundle is denoted by TM, and its dual by T^*M. More complicated bundles are obtained by taking tensor products of these. Of particular importance is the tensor bundle $T^*M \otimes TM$, the space of linear maps of TM. This space is naturally isomorphic to its dual $TM \otimes T^*M$, the space of linear maps from T^*M to itself: A map A in $T^*M \otimes TM$ is associated with its adjoint, denoted $A^\dagger$. The identity map in either space is denoted Id. This gives rise to a natural inner product on $T^*M \otimes TM$, independent of any metric on TM: For maps A and B in $T^*M \otimes TM$, we can take the product $A(B^\dagger)$ given by the duality pairing. In particular $|A|^2 = A(A^\dagger)$ and $\mathrm{tr}A = A(\mathrm{Id})$ are the squared modulus and trace of A. A choice of metric g on TM gives an isomorphism between TM and T^*M, which allows this inner product to be extended to the other tensor bundles. g also gives a natural correspondence between other tensor bundles—in particular, a tensor $\mathcal{T} \in T^*M \otimes T^*M$ naturally corresponds to a map $g^*\mathcal{T} \in T^*M \otimes TM$, by the relation $g(u, g^*\mathcal{T}(v)) = \mathcal{T}(u, v)$ for every u and v in TM.

A metric also has an associated Levi-Civita connection ∇, which can be used to define tensorial derivatives for any tensor by the Leibnitz rule:

$$\nabla(A \otimes B) = (\nabla A) \otimes B + A \otimes (\nabla B). \tag{2–1}$$

Suppose T is a covariant k-tensor (equivalently, a multilinear function of k vectors).

Then the derivative is a covariant $(k+1)$-tensor defined as follows:

(2–2)
$$\nabla T(u, v_1, \ldots, v_k) = d_u T(v_1, \ldots, v_k) - \sum_{i=1}^{k} T(v_1, \ldots, v_{i-1}, \nabla_u v_i, v_{i+1}, \ldots, v_k)$$

Repeating this procedure gives higher tensorial derivatives. In particular, the second tensorial derivative is called the *Hessian.* In the case of functions, it is defined as follows:

(2–3)
$$\mathrm{Hess}_\nabla f(u, v) = d_u d_v f - d_{\nabla_u v} f.$$

The metric g and connection ∇ have an associated Riemann curvature tensor R, defined as follows for vectors u, v, and w:

(2–4)
$$R(u, v, w) = \left(\nabla_v \nabla_u - \nabla_u \nabla_v - \nabla_{[u,v]}\right) w.$$

This definition leads to the following rules for the commutation of covariant derivatives:

(2–5)
$$\begin{aligned}(\mathrm{Hess}_\nabla \mathcal{T})(u, v, w_1, \ldots, w_k) = &(\mathrm{Hess}_\nabla \mathcal{T})(v, u, w_1, \ldots, w_k) \\ &+ \sum_{i=1}^{k} \mathcal{T}(w_1, \ldots, w_{i-1}, R(u, v, w_i), w_{i+1}, \ldots, w_k)\end{aligned}$$

for any tensor $\mathcal{T} \in \otimes^k T^* M$.

The objects of interest in this section of the thesis are families of smooth hypersurfaces of codimension one in Euclidean space $\mathbb{R}^{n+1}$. We will consider each hypersurface as the image of a smooth immersion φ from a smooth manifold M^n to $\mathbb{R}^{n+1}$; for the purposes of analysis we will work directly with the family of immersions $\varphi : M^n \times [0, T) \to \mathbb{R}^{n+1}$, keeping in mind the invariance of the geometry under diffeomorphisms of the manifold M^n.

We denote the standard metric on Euclidean space by angle brackets $\langle \ldots, \ldots \rangle$, and the standard connection by D. Each immersion φ_t induces on M a metric g and a connection ∇ corresponding to these (the dependence of the various geometric quantities on time will not be made explicit):

$$g(u,v) = \langle T\varphi(u), T\varphi(v)\rangle \tag{2–6}$$

$$\nabla_u v = T_x\varphi^{-1}\left(\pi_x\left(D_{T\varphi(u)}T\varphi(v)\right)\right)$$

for all vector fields u and v in TM^n. Here $T_x\varphi$ is the derivative of φ at x, and π_x is the orthogonal projection of $\mathbb{R}^{n+1}$ onto the image of $T_x\varphi$.

The curvature of the hypersurface is given by the normal component of the connection on $\mathbb{R}^{n+1}$, called the second fundamental form $I\!I \in T^*M \otimes T^*M$, which is symmetric with respect to the metric g:

$$I\!I(u,v) = -\langle D_{T\varphi(u)}T\varphi(v), \nu\rangle \tag{2–7}$$

for all u and v in T_xM^n. The Codazzi and Gauss equations follow from the definitions of the second fundamental form and the Riemann tensor:

$$\nabla I\!I(u,v,w) = \nabla I\!I(v,u,w) \tag{2–8}$$

$$g(R(u,v,w),z) = I\!I(u,w)I\!I(v,z) - I\!I(v,w)I\!I(u,z) \tag{2–9}$$

for all u, v, w and z in TM^n.

The Weingarten map (or Weingarten curvature) $\mathcal{W} : TM^n \to TM^n$ gives the rate of change in the direction of the normal along the surface:

$$\mathcal{W}(u) = T\varphi^{-1}\left(T\nu(u)\right) \tag{2–10}$$

for all u in T_xM^n, where $T\nu : TM \to TS^n \simeq T\varphi(TM)$ is the derivative of ν. This is related to the second fundamental form as follows:

$$I\!I(u,v) = g\left(\mathcal{W}(u), v\right). \tag{2–11}$$

The eigenvalues of the Weingarten map (which are the eigenvalues of the second fundamental form with respect to the metric g) are called the *principal curvatures* of φ, and are denoted $\lambda_1, \ldots, \lambda_n$.

We are concerned particularly with the case where the immersions are strictly locally convex (the second fundamental form is positive definite everywhere). This makes the Gauss map $\nu : M^n \to S^n$ everywhere nondegenerate (from (2–10) and (2–11)), and therefore a diffeomorphism. Let $\bar{g}$ and $\bar{\nabla}$ be the standard metric and connection on S^n. The Weingarten map can be used to relate $\bar{g}$ to the metric g:

$$\bar{g}(T\nu(u), T\nu(v)) = g\left(\mathcal{W}(u), \mathcal{W}(v)\right). \tag{2–12}$$

The connections ∇ and $\bar{\nabla}$ are also related:

$$T\nu^{-1}\left(\bar{\nabla}_{T\nu(u)}T\nu(v)\right) - \nabla_u v = \mathcal{W}^{-1}\left(\nabla\mathcal{W}(u,v)\right). \tag{2–13}$$

It is often convenient to parametrise a convex hypersurface by the Gauss map. The hypersurface can be conveniently described using the *support function*, which is a real function defined on the sphere S^n follows:

$$s(z) = \langle z, \varphi\left(\nu^{-1}(z)\right)\rangle \tag{2–14}$$

for all z in S^n. If the support function is known, the hypersurface is given as the boundary of the convex region $\bigcap_{z \in S^n}\{y \in \mathbb{R}^{n+1} : \langle y, z\rangle \le s(z)\}$. Further details of this approach to the description of convex hypersurfaces will be given in section II of the thesis. The second fundamental form can be calculated directly from the support function as follows:

$$I\!I\left(T\nu^{-1}(u), T\nu^{-1}(v)\right) = \left(\mathrm{Hess}_{\bar{\nabla}} s + s\bar{g}\right)(u,v). \tag{2–15}$$

The support function provides some useful means of describing the general shape of a convex hypersurface—the *width function* is defined on S^n by the equation $w(z) = s(z) + s(-z)$. This gives the separation of the tangent planes at points with opposing normal directions. The maximum and minimum widths are denoted w_+ and w_- respectively. Note that w_+ is the diameter of $\varphi(M)$. It will be shown in chapter 5 how these quantities may be related to the *inradius* and *circumradius* ρ_- and ρ_+, which are defined as follows:

(2–16)

$$\rho_+ = \inf\{r : B_r(y) \text{ encloses } \varphi(M) \text{ for some } y \in \mathbb{R}^{n+1}\}$$

$$\rho_- = \sup\{r : B_r(y) \text{ is enclosed by } \varphi(M) \text{ for some } y \in \mathbb{R}^{n+1}\}$$

where $B_r(y)$ is the ball of radius r with centre at y.

An alternative approach for a hypersurface which is star-shaped about the origin is to parametrise as a graph over the unit sphere—if such a hypersurface is given by an immersion $\varphi : M^n \to \mathbb{R}^{n+1}$, we have a nondegenerate map $\bar{\pi}$ from M^n to S^n, given by $\bar{\pi}(x) = \frac{\varphi(x)}{|\varphi(x)|}$. This is the restriction of the radial projection of $\mathbb{R}^{n+1}$ onto S^n. The radial distance function, defined for all z in S^n by $r(z) = \left|\varphi\left(\bar{\pi}^{-1}(z)\right)\right|$, contains all information about the hypersurface. The following equations hold, where $\beta = \frac{1}{\sqrt{r^2+\left|\bar{\nabla} r\right|^2}}$:

(2–17)

$$g(T\bar{\pi}^{-1}u, T\bar{\pi}^{-1}v) = r^2\bar{g}(u.v) + \bar{\nabla}_u r.\bar{\nabla}_v r$$

(2–18)

$$I\!I(T\bar{\pi}^{-1}u, T\bar{\pi}^{-1}v) = -\beta\left(r\mathrm{Hess}_{\bar{\nabla}} r - r^2\bar{g} - 2\bar{\nabla}r \otimes \bar{\nabla}r\right)(u, v)$$

(2–19) $$\mathcal{W}(T\bar{\pi}^{-1}u) = -\left(r^{-1}\bar{g}^*\left(\bar{\nabla}\left(\beta\bar{\nabla}r\right)\right) + \beta\mathrm{Id}\right)(u)$$

(2–20)

$$T\bar{\pi}\left(\nabla_{T\bar{\pi}^{-1}u}T\bar{\pi}^{-1}v\right) - \bar{\nabla}_u v = -\frac{\beta}{r}I\!I(T\bar{\pi}^{-1}u, T\bar{\pi}^{-1}v)\bar{\nabla}r + \frac{1}{r}\left((\bar{\nabla}_v r)u + (\bar{\nabla}_u r)v\right).$$

I will conclude this chapter with some remarks on symmetric functions, which will be useful in dealing with the speed function F. Suppose f is a symmetric function of n variables $\lambda_1, \ldots, \lambda_n$, defined on the positive cone Γ_+ of $\mathbb{R}^n$, defined by $\Gamma_+ := \{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i > 0, i = 1, \ldots, n\}$.

Lemma 2–21. *Fix i and j in $\{1, \ldots, n\}$. Suppose f is concave (convex), and $\lambda \in \Gamma_+$ is such that $\lambda_i > \lambda_j$. Then $\frac{\partial f}{\partial \lambda_i} - \frac{\partial f}{\partial \lambda_j}$ is negative (positive).*

Proof : Let η be the vector $\dfrac{\partial}{\partial \lambda_i} - \dfrac{\partial}{\partial \lambda_j}$ in $\mathbb{R}^n$. Define a curve $\tilde{\lambda} : [0,1] \to \Gamma_+$ by $\tilde{\lambda}(r) = \lambda + r\frac{\lambda_j - \lambda_i}{2}\eta$. Note that $\frac{\partial f}{\partial \lambda_i} - \frac{\partial f}{\partial \lambda_j} = D_\eta f$, and by symmetry we have $D_\eta f\left(\tilde{\lambda}(1)\right) = 0$. Integrating along the curve gives the following:

$$\left(\frac{\partial f}{\partial \lambda_i} - \frac{\partial f}{\partial \lambda_j}\right)(\lambda) = \frac{\lambda_i - \lambda_j}{2}\int_0^1 D_\eta D_\eta f\left(\tilde{\lambda}(r)\right) dr \tag{2–22}$$

which is negative (positive) as required. ∎

Given a symmetric function f as above, one can define a function $F : S_+ \to \mathbb{R}$ by $F(A) = f(\lambda(A))$ for any $A \in S_+$, where $\lambda(A)$ gives the eigenvalues of A.

Lemma 2–23. *Let f and F be defined as above.*

(1). If f is smooth, then F is smooth.

(2). If $\frac{\partial f}{\partial \lambda_i} > 0$ for $i = 1, \ldots, n$ at some point $A \in T^*M \otimes TM$ then the tensor $\dot{F} \in TM \otimes T^*M$ defined by $\dot{F}(B) = D_B F$ is positive definite.

(3). If f is convex (concave), then F is convex (concave).

Proof : (See also [**CNS1**]). f can be written as a smooth function of the elementary symmetric functions $e_1, e_2, \ldots, e_n$, defined by

$$e_k(\lambda) = \frac{1}{\binom{n}{k}} \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_k}.$$

Thus F can be written as a smooth function of the corresponding functions $E_k(A) = \mathrm{tr}(A^{(k)})$, where $A^{(k)}$ is the map of $\otimes^k TM$ given by

$$A^{(k)}(u_1, \ldots, u_k) = \frac{1}{\binom{n}{k}} \sum_{\sigma \in S(k)} (-1)^{\mathrm{sgn}(\sigma)} \left(A\left(u_{\sigma(1)}\right), \ldots, A\left(u_{\sigma(k)}\right)\right).$$

Here $S(k)$ is the group of permutations of k objects. These functions are smooth, and therefore so is F.

The second claim is clearly preserved by any similarity transformation. Perform such a transformation to make A diagonal. Then $\dot{F} = \mathrm{diag}\left(\frac{\partial f}{\partial \lambda_1}, \ldots, \frac{\partial f}{\partial \lambda_n}\right)$.

The last assertion can be proved as follows: Clearly convexity (concavity) is preserved by a similarity transformation, so it is sufficient to prove that F is convex (concave) at any diagonal A in S_+. Calculating directly at such a point, we find:

$$\ddot{F}(\xi, \xi) = \sum_{p,q} \frac{\partial^2 f}{\partial \lambda_p \partial \lambda_q} \xi_p^p \xi_q^q + \sum_{p \neq q} \frac{\frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_q}}{\lambda_p - \lambda_q} \left(\xi_q^p\right)^2 \tag{2–24}$$

where $\ddot{F} \in TM \otimes T^*M \otimes TM \otimes T^*M$ is the second derivative of F at the point $A \in S_+$. The result now follows from lemma (2–21). ∎

3. The Evolution Equations

I will begin by specifying the precise conditions required of the speed and the initial hypersurface for theorem (1–2):

Conditions 3–1. *The speed function F and the initial immersion φ_0 are assumed to satisfy the following conditions:*

(1). $F(\mathcal{W}) = f(\lambda(\mathcal{W}))$ where $\lambda(\mathcal{W})$ gives the eigenvalues of $\mathcal{W}$, and f is a smooth symmetric function defined on the positive cone Γ_+.

(2). f is strictly increasing in each argument: $\frac{\partial f}{\partial \lambda_i} > 0$ for $i = 1, \ldots, n$ at every point in Γ_+.

(3). f is homogeneous of degree one: $f(k\lambda) = kf(\lambda)$ for any positive $k \in \mathbb{R}$.

(4). f is strictly positive on Γ_+, and $f(1, \ldots, 1) = 1$.

(5). One of the following holds:

(i). f is convex;

or (ii). f is concave and either

(a). $n = 2$;

(b). f approaches zero on the boundary of Γ_+;

or (c). $\sup_{t=0}\left(\frac{H}{F}\right) < \liminf_{\lambda \to \partial\Gamma_+}\left(\frac{\sum \lambda_i}{f(\lambda)}\right)$.

The first condition ensures that the evolution equation is isotropic—that is, invariant under rotations in $\mathbb{R}^{n+1}$. The second condition makes the equations into a parabolic system: Lemma (2–23) shows that the derivatives of F with respect to the components of the Weingarten map form a positive definite map. This gives strict parabolicity in the normal direction, with degeneracy in all tangential directions. The condition (4) is simply a normalisation condition, which can always be satisfied by rescaling time—conditions (2) and (3) together imply that f is strictly positive on Γ_+. These conditions allow results of similar generality to those in the papers [**Ge**], [**Hu4**], [**U1**], and [**U2**] which deal with the expansion of hypersurfaces by speeds which are homogeneous of degree -1 in the principal curvatures. Similar conditions have also been used for elliptic equations—see for example [**CNS1**], [**CNS2**], [**CNS3**].

There are many examples of functions f which satisfy these conditions: The mean curvature $H = \frac{1}{n}\sum_{i=1}^{n}\lambda_i$ is convex, and so satisfies (5(i)); the nth root of the Gauss curvature $K^{\frac{1}{n}} = (\lambda_1 \ldots \lambda_n)^{\frac{1}{n}}$ is concave, and zero on the boundary of Γ_+); this is condition (5(ii)b). Other suitable classical curvatures include those of the form $\left(\frac{K}{e_k}\right)^{\frac{1}{n-k}}$, where e_k is the k-th elementary symmetric function, for $k = 0, \ldots, n-1$. These are all concave and zero on the boundary on Γ_+. The power means, $H_r = \left(\frac{1}{n}\sum \lambda_i^r\right)^{\frac{1}{r}}$, are convex for $r \geq 1$, and concave for $r \leq 1$; they are zero on the boundary of Γ_+ for all $r < 0$. A particular case of interest here is the harmonic mean curvature ($r = -1$).

Note that condition (5(ii)c) means that any concave homogeneous symmetric function f can be used, provided the initial hypersurface is sufficiently pinched pointwise—that is, provided φ_0 satisfies $\lambda_{\max}(x) < C\lambda_{\min}(x)$ for all x in M, where C depends only on f. A somewhat more general condition will also suffice—see the remarks after theorem (4–1). In particular, a class of flows of some interest is $f = (S_k(\lambda))^{\frac{1}{k}}$ for $k = 1, \ldots, n$. If $n \neq 2$ and $k \neq 1$ or n, then we require a pinching condition for the initial immersion φ_0. This pinching condition is identical to one that appears in [**Ch2**] for the case $k = 2$, the square root of the scalar curvature. Such a condition is also required for the power means with r in the range $(0, 1)$. It is not clear, however, that this condition is necessary in any of these cases—as shown in part III of this thesis, all the examples just mentioned will serve to contract convex hypersurfaces to points, and it is tempting to conjecture that the convergence of rescaled solutions to spheres may follow by some modified argument.

The degeneracy of the system of equations (1–1) is related to its invariance under diffeomorphisms of M^n. Short time existence can be proved in a several ways - by writing the surfaces M_t as graphs over the initial surface M_0, or over a sphere centred at a point contained within the initial surface, or by considering the parametrisation by the Gauss map (see section II for more details of this technique). These methods all serve to fix the parametrisation of the hypersurfaces, breaking the diffeomorphism symmetry. The spherical graph technique will be important in the analysis to follow:

Lemma 3–2. *There is a one-to-one correspondence between smooth solutions φ to equation (1–1) which are star-shaped about the origin, and smooth positive solutions r to the following scalar parabolic equation on the sphere:*

$$\text{(3–3)} \qquad \begin{aligned} \frac{\partial}{\partial t} r(z,t) &= \mathcal{F}\left(\frac{1}{\beta r^2}\bar{g}^*\left(\bar{\nabla}\left(\beta\bar{\nabla}r\right)\right) - \frac{\mathrm{Id}}{r}\right) \\ r(z,0) &= \left|\varphi_0\left(\bar{\pi}_0^{-1}(z)\right)\right| \end{aligned}$$

where $\bar{\pi}_t : M^n \to S^n$ is the projection given by $\bar{\pi}_t(x) = \frac{\varphi(x,t)}{|\varphi(x,t)|}$, and $\mathcal{F} : S_- \to \mathbb{R}$ is defined by $\mathcal{F}(A) = -F(-A)$ where $S_- \subset TS^n \otimes T^*S^n$ is the negative cone $\{\mathcal{Z} : -\mathcal{Z} \in S_+\}$.

Proof : If φ is starshaped about the origin, then the projection $\bar{\pi}$ is a diffeomorphism. For a time t_0 and a point z in S^n, let $y = \bar{\pi}_{t_0}^{-1}(z)$. Then $\varphi(y) = r(z)z$. Resolving the evolution equation (1–1) into components normal and perpendicular to z, we have the following equations at time t_0:

$$\frac{\partial}{\partial t} r(\bar{\pi}(y)) = -\beta r F(\bar{\pi}(y))$$
$$\frac{\partial}{\partial t}(\bar{\pi}(y)) = \frac{\beta}{r} F \bar{\nabla} r(\bar{\pi}(y))$$

Now the evolution equation for r at the point z is given by:

$$\begin{aligned}\frac{\partial}{\partial t} r(z) &= \frac{\partial}{\partial t} r(\bar{\pi}(y)) - \bar{g}(\bar{\nabla} r, \frac{\partial}{\partial t}(\bar{\pi}(y))) \\ &= -\frac{F\beta}{r}(r^2 + |\bar{\nabla} r|^2) \\ &= -\frac{F}{\beta r} \qquad (3\text{–}4)\end{aligned}$$

and the equation (3–3) follows from the homogeneity of F and the equation (2–19).

Conversely, a smooth solution to (3–3) gives rise to a solution of (1–1) as follows: Define a family of immersions $\hat{\varphi} : S^n \times [0, T) \to \mathbb{R}^{n+1}$ by $\hat{\varphi}_t(z) = r(z)z$, and let $\psi : M^n \times [0, T) \to S^n$ be the diffeomorphisms obtained by solving the following ordinary differential equation:

$$\frac{\partial}{\partial t}\psi_t(x) = \frac{\beta F \bar{\nabla} r}{r} \qquad (3\text{–}5)$$
$$\psi_0(x) = \bar{\pi}(x)$$

where F is evaluated at the map $\mathcal{W}(\bar{\pi}^{-1}(z))$ given by equation (2–19). Then define $\varphi : M^n \times [0, T) \to \mathbb{R}^{n+1}$ by $\varphi_t(x) = \hat{\varphi}_t(\psi_t(x))$. Then φ is a solution to (1–1). The correspondence is one-to-one since the solution to (3–5) is unique. ∎

Corollary 3–6. *For any strictly convex, smooth initial immersion φ_0 there exists a unique smooth solution φ_t to equation* (1–1) *on some time interval* $[0, T)$.

Proof : By the previous lemma, a solution to (1–1) is given in terms of a solution to a strictly parabolic scalar equation on the sphere S^n. Uniqueness and short-time existence follow by standard parabolic theory. ∎

In order to study the behaviour of solutions, it is useful to know how the metric and curvature of the immersions evolve. The evolution equations governing these quantities can be deduced from equation (1–1):

Theorem 3–7. *The following evolution equations hold for any solution to equation* (1–1)*:*

$$\frac{\partial}{\partial t} g = -2F\mathit{I\!I} \tag{3–8}$$

$$\frac{\partial}{\partial t} \nu = T\varphi(\nabla F) \tag{3–9}$$

$$\frac{\partial}{\partial t} \mathit{I\!I} = \mathrm{Hess}_{\nabla} F - F\bar{g} \tag{3–10}$$

$$\frac{\partial}{\partial t} \mathcal{W} = g^* \left(\mathrm{Hess}_{\nabla} F\right) + F\mathcal{W}^2 \tag{3–11}$$

$$\frac{\partial}{\partial t} F = \dot{F} g^* \left(\mathrm{Hess}_{\nabla} F\right) + F\dot{F}\left(\mathcal{W}^2\right). \tag{3–12}$$

Proof : Since the metric $\langle \ldots, \ldots \rangle$ on $\mathbb{R}^{n+1}$ is independent of time,

$$\begin{aligned}
\frac{\partial}{\partial t} g(u,v) &= \frac{\partial}{\partial t} \langle D_u\varphi, D_v\varphi\rangle \\
&= \langle \frac{\partial}{\partial t} D_u\varphi, D_v\varphi\rangle + \langle D_u\varphi, \frac{\partial}{\partial t} D_v\varphi\rangle \\
&= \langle D_u(-F\nu), D_v\varphi\rangle + \langle D_u\varphi, D_v(-F\nu)\rangle \\
&= -F\langle D_u\nu, D_v\varphi\rangle - F\langle D_u\varphi, D_v\nu\rangle \\
&= -Fg\left(\mathcal{W}(u), v\right) - Fg\left(u, \mathcal{W}(v)\right) \\
&= -2F\mathit{I\!I}(u,v)
\end{aligned}$$

using the definition of $\mathcal{W}$ and the identity (2–11).

Since $\frac{\partial}{\partial t}\nu$ is in $T_\nu S^n$, $T_x^{-1}\varphi\left(\frac{\partial}{\partial t}\nu\right)$ is in $T_x M^n$; in particular:

$$g\left(T\varphi^{-1}\left(\frac{\partial}{\partial t}\nu\right), u\right) = \langle \nu, \frac{\partial}{\partial t} D_u\varphi\rangle = \langle \nu, D_u(-F\nu)\rangle = -D_u F.$$

The evolution of $I\!I$ can be calculated from the definition (2–7):

$$\begin{aligned}\frac{\partial}{\partial t}I\!I(u,v) &= -\frac{\partial}{\partial t}\langle D_{T\varphi(u)}T\varphi(v),\nu\rangle\\ &= d_u d_v F + F\langle d_u T\varphi(\mathcal{W}(v)),\nu\rangle - \langle d_u d_v\varphi, T\varphi(\nabla F)\rangle\\ &= d_u d_v F - d_{\nabla_u v}F - F\langle T\varphi(\mathcal{W}(v)), T\varphi(\mathcal{W}(u))\rangle\\ &= \mathrm{Hess}_\nabla F - Fg(\mathcal{W}(u),\mathcal{W}(v))\end{aligned}$$

using the definition of the Weingarten map. The relation (2–11) gives the evolution of the Weingarten map by combining equations (3–8) and (3–10):

$$\begin{aligned}g\left(u,\frac{\partial}{\partial t}\mathcal{W}(v)\right) &= \frac{\partial}{\partial t}I\!I(u,v) - \frac{\partial}{\partial t}g(u,\mathcal{W}(v))\\ &= \frac{\partial}{\partial t}I\!I(u,v) + 2F\bar{g}(u,v)\end{aligned}$$

which implies equation (3–11).

Finally,

$$\begin{aligned}\frac{\partial}{\partial t}F &= \frac{\partial}{\partial t}F(\mathcal{W})\\ &= \dot{F}\left(\frac{\partial}{\partial t}\mathcal{W}\right)\end{aligned}$$

and the result (3–12) follows. ∎

The expression (3–11) for the evolution of the Weingarten map, while encouragingly simple, is not in the form of a parabolic equation. Some manipulation of the first term gives the following useful expression, which will allow us to apply the parabolic maximum principle.

Lemma 3–13.

$$\frac{\partial}{\partial t}\mathcal{W} = \dot{F}\left(g^*\left(\mathrm{Hess}_\nabla\mathcal{W}\right)\right) + \dot{F}\left(\mathcal{W}^2\right)\mathcal{W} + g^*\ddot{F}\left(\nabla\mathcal{W},\nabla\mathcal{W}\right).$$

Proof: Note that the definitions of $\dot{F}$ and $\ddot{F}$ allow us to write equation (3–11) as follows:

$$\frac{\partial}{\partial t}\mathcal{W}(u) = (g^*\mathrm{Hess}_\nabla\mathcal{W})(u,\dot{F}) + g^*\ddot{F}(\nabla\mathcal{W},\nabla\mathcal{W})(u) + F\mathcal{W}^2(u). \tag{3–14}$$

The result follows from a form of Simons's identity, which is a consequence of the Gauss and Codazzi equations:

$$\begin{aligned}\mathrm{Hess}_\nabla I\!I(u,v,w,z) =& \mathrm{Hess}_\nabla I\!I(w,z,u,v) + I\!I(u,v)\bar{g}(w,z) - I\!I(w,z)\bar{g}(u,v)\\ &+ I\!I(u,z)\bar{g}(v,w) - I\!I(v,w)\bar{g}(u,z).\end{aligned} \tag{3–15}$$

By applying the correspondence g^*, contracting with $\dot{F}$ and acting on u, the following equation is obtained:

$$\left(g^*\mathrm{Hess}_\nabla\mathcal{W}(u,\dot{F})\right) = \left(g^*\mathrm{Hess}_\nabla\mathcal{W}(\dot{F},u)\right) + \mathcal{W}(u)\dot{F}\left(\mathcal{W}^2\right) - \dot{F}(\mathcal{W})\,\mathcal{W}^2(u).$$

Combining this with (3–14) yields the desired result using the Euler homogeneity equation, since $\dot{F}(\mathcal{W}) = \sum_{i=1}^n \lambda_i \frac{\partial f}{\partial \lambda_i} = f = F$. It is here that we require the homogeneity of degree one of the speed. ∎

It is convenient to define an elliptic operator $\mathcal{L}$ by $\mathcal{L}(\psi) = \dot{F}(g^*\mathrm{Hess}_\nabla\psi)$ for any function ψ on M^n. The leading order terms of equations (3–12) and (3–13) are then given by $\mathcal{L}$.

When using the spherical graph parametrisation, it is convenient to have the following result in order to calculate evolution equations:

Lemma 3–16. *Suppose σ is a scalar quantity which evolves under* (1–1) *according to the following evolution equation:*

$$\frac{\partial}{\partial t}\sigma(x,t) = \mathcal{L}\sigma(x,t) + Q(x,t).$$

Then if $\bar{\sigma}(z,t) = \sigma(\bar{\pi}^{-1}(z),t)$, *the following evolution equation holds for* $\bar{\sigma}$ *under equation* (3–3)*:*

$$\frac{\partial}{\partial t}\bar{\sigma}(z,t) = \bar{\mathcal{L}}\bar{\sigma}(z,t) + Q(\bar{\pi}^{-1}(z),t) - \dot{F}g^*(\bar{\nabla}\bar{\sigma}\otimes\bar{\nabla}r + \bar{\nabla}r\otimes\bar{\nabla}\bar{\sigma})$$

where g *is given by equation* (2–17) *and* $\bar{\mathcal{L}} = \dot{F}\bar{g}^*\mathrm{Hess}_{\bar{\nabla}}$.

Proof : This follows easily by combining the expressions for the ordinary differential equation (3–5) and the difference in the connections (2–20). ∎

4. Preserving Convexity

In this chapter it is shown that strict convexity is preserved under the evolution equation (1–1), and that the principal curvatures satisfy a pointwise pinching condition, for as long as the solution exists. This is proved by applying the parabolic maximum principle to the evolution equation for an appropriate scaling-invariant curvature quantity. The details of this calculation differ slightly for the different conditions allowed in (3–1).

Theorem 4–1. *Let φ be a solution to equation (1–1), where φ_0 and F satisfy the conditions (3–1). Then there exist $\epsilon > 0$ and $C_1 < \infty$ such that as long as the solution exists, the following estimates hold:*

$$\lambda_i(x) \geq \epsilon \tag{4–2}$$

$$\frac{\lambda_i(x)}{\lambda_j(x)} \leq C_1 \tag{4–3}$$

for every x in M^n and $1 \leq i, j \leq n$.

Proof : Equation (3–13) implies that the infinum of F over M^n is increasing, by the parabolic maximum principle. Now calculate the evolution equation for $\frac{\mathcal{W}}{F}$ from the equations (3–12) and (3–13):

$$\frac{\partial}{\partial t}\left(\frac{\mathcal{W}}{F}\right) = \mathcal{L}\left(\frac{\mathcal{W}}{F}\right) + 2F^{-1}\dot{F}\left(\nabla F, \nabla\left(\frac{\mathcal{W}}{F}\right)\right) + F^{-1}g^{*}\ddot{F}\left(\nabla\mathcal{W}, \nabla\mathcal{W}\right). \tag{4–4}$$

Consider now the case where f is a convex function of the principal curvatures. Then lemma (2–23) ensures that the last term in (4–4) is positive. The parabolic maximum principle then ensures that the infinum over the unit ball in TM of $\frac{\mathcal{W}}{F}$ is increasing in time. Consequently there is some number C such that for all x in

M^n and all t for which the solution exists, $\lambda_{\min}(x) > CF(x)$. Since the infinum of F is increasing, the claim (4–2) follows immediately. The convexity of F and condition (4) in (3–1) imply that $F \geq H \geq \lambda_{\max}$. Therefore as required, there is some constant C_1 such that $\frac{\lambda_{\max}(x)}{\lambda_{\min}(x)} \leq C_1$.

Next consider the concave case. First observe that conditions (5(ii)a) and (5(ii)b) simply imply (5(ii)c), so only this last case need be considered. Using the equation (4–4) above, one can deduce the evolution equation for $\frac{H}{F}$:

$$\text{(4–5)} \qquad \frac{\partial}{\partial t}\left(\frac{H}{F}\right) = \mathcal{L}\left(\frac{H}{F}\right) + 2F^{-1}\dot{F}\left(\nabla F, \nabla\left(\frac{H}{F}\right)\right) + \frac{1}{nF}\mathrm{tr}_g\ddot{F}(\nabla \mathcal{W}, \nabla \mathcal{W})$$

Since f is concave, lemma (2–23) shows that the last term in this equation is negative, and the parabolic maximum principle implies that the supremum of $\frac{H}{F}$ is decreasing. The assumption (5(ii)c) guarantees that the region $\{\frac{\sum\lambda}{f(\lambda)} \leq \sup_{t=0}\frac{H}{F}\}$ does not touch the boundary of the positive cone Γ_+ except at the origin. Since $\frac{H}{F}$ is homogeneous of degree zero, this implies a bound of the form (4–3). The estimate (4–2) follows. ∎

Remark : In the concave case, H can be replaced in condition (5(ii)c) by any function G which is homogeneous of degree one and convex. This may be useful in some cases where F is not strictly concave.

Corollary 4–6. *There exist constants $\underline{C}$ and $\overline{C}$ such that*

$$\text{(4–7)} \qquad \underline{C}\mathrm{Id} \leq \dot{F} \leq \overline{C}\mathrm{Id}.$$

Proof : $\dot{F}$ is smooth and strictly positive in Γ_+. Note that $\dot{F}(\lambda) = \dot{F}(\frac{\lambda}{|\lambda|})$, and $\frac{\lambda}{|\lambda|}$ is in the compact subset of Γ_+ given by $\{|\lambda| = 1, \lambda_{\max} \leq C_1\lambda_{\min}\}$. $\dot{F}$ has a positive lower bound and a finite upper bound on this set. ∎

5. The Consequences of Pinching

This chapter proves a simple geometrical consequence of the pointwise pinching estimate of theorem (4–1). This result holds for arbitrary convex compact hypersurfaces, and does not depend in any way on the evolution equation (1–1).

Theorem 5–1. *Let $\varphi : M^n \to \mathbb{R}^{n+1}$ be a smooth, strictly convex immersion of the compact manifold M^n, and suppose that φ satisfies the following pointwise pinching estimate for some $C_1 < \infty$:*

$$\lambda_{\max}(x) \leq C_1 \lambda_{\min}(x)$$

for every x in M^n. Then the following estimate holds:

$$w_+ \leq C_1 w_-. \tag{5–2}$$

Proof : First note that the eigenvalues of the map $A = \bar{g}^* \mathrm{Hess}_{\bar{\nabla}} s + \mathrm{Id} s$ also satisfy a pinching condition, with the same constant C_1:

$$A(u) = T\nu(\mathcal{W}^{-1}(T\nu^{-1}(u))). \tag{5–3}$$

By definition of the support function s, there exist points z_+ and z_- in S^n such that $w_+ = s(z_+) + s(-z_+)$ and $w_- = s(z_-) + s(-z_-)$. Let Σ be any totally geodesic 2–sphere in S^n which contains both z_+ and z_-.

Define two spherical coordinate systems $(\phi_\pm, \theta_\pm)$ on Σ: $\phi_\pm(z) = \sin^{-1}\langle z, z_\pm\rangle$, and $\theta_\pm$ is the angle around a great circle perpendicular to $z_\pm$. The following calculation gives expressions for the widths of the hypersurface $\varphi(M^n)$:

$$\begin{aligned}\int_\Sigma I\!I\,(\partial\phi_\pm, \partial\phi_\pm)\, d\mu_\Sigma &= \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \left(\bar{\nabla}_{\phi_\pm}\bar{\nabla}_{\phi_\pm} s + s\right)\cos\phi_\pm d\phi_\pm d\theta_\pm \\ &= 2\pi\left(s(z_\pm) + s(-z_\pm)\right)\end{aligned}$$

after integrating by parts twice. Note that $\partial\phi_+$ and $\partial\phi_-$ have unit length almost everywhere with respect to $\bar{g}$, so $\bar{g}\left(A\left(\partial\phi_+\right),\partial\phi_+\right) \leq C_1\bar{g}\left(A\left(\partial\phi_-\right),\partial\phi_-\right)$ almost everywhere. ∎

It is useful to relate this estimate to the inradius and circumradius $\rho_\pm$:

Lemma 5–4. *For any compact, convex hypersurface, the following estimates hold:*

(5–5)
$$\rho_+ \leq \frac{w_+}{\sqrt{2}}$$
$$\rho_- \geq \frac{w_-}{n+2}$$

and consequently we have $\rho_+ \leq C_2\rho_-$ for some constant C_2.

Proof: Let Σ be a sphere of smallest radius which encloses $\varphi(M)$, and assume it has centre at the origin. Let $S = \Sigma \cap \varphi(M)$, and assume that z_0 and z_1 are two points in S which maximise the distance $|z_0 - z_1|$. Clearly the angle between z_0 and z_1 is obtuse, since otherwise Σ could be moved to strictly contain $\varphi(M)$, contradicting the assumption that Σ has smallest possible radius. Then the distance from z_0 to z_1 is a lower bound for the maximum width w_+, and is at least $\sqrt{2}$ times the radius of Σ, or $\sqrt{2}\rho_+$.

Now let Σ be a sphere of largest radius enclosed by $\varphi(M)$, and choose the origin at the centre of Σ. Let $S = \Sigma \cap \varphi(M)$. One can show that there is a nonempty set of points $P \subset S$ such that $P\backslash\{z\}$ is linearly independent for any z in P, and such that there is a positive linear combination of the elements of P with value zero—if this were not the case, then the convex hull of S could not contain the origin, and so Σ could be moved slightly to become properly contained by $\varphi(M)$. Let E be the smallest affine subspace of $\mathbb{R}^{n+1}$ which contains the set P. Note that E has dimension $k-1$, where P has k elements. Let $\bar{S}$ be the simplex $\{y \in E : \langle y, z\rangle \leq s(z) \text{ for all } z \in P\}$. By convexity, $\bar{S}$ contains the projection of $\varphi(M)$ onto E. Hence the minimum width of $\varphi(M)$ is no greater than the minimum width of $\bar{S}$, which is the shortest altitude of $\bar{S}$. This is bounded by the altitude of a regular simplex inscribed by Σ in E, or $k\rho_-$. Since E has dimension at most $n+1$, the result follows. ∎

6. Convergence to a Point

In this chapter the first part of theorem (1–2) is proved—it is shown that the solution remains smooth on a finite time interval $[0,T)$, and converges uniformly to a constant p in $\mathbb{R}^{n+1}$ as the final time is approached.

Theorem 6–1. *Suppose φ is a smooth solution to the equation* (1–1)*, such that the inner radius ρ_- is not less than some positive value ρ_0 on the time interval $[0,t_0)$. Then for some positive δ, φ extends to the time interval $[0,t_0+\delta)$.*

Proof : It is sufficient to show that for each positive k there are C^k bounds which hold up to the time t_0. Then one has convergence in C^∞ to a smooth strictly convex hypersurface at time t_0—this can be seen by considering the convergence of the support function, for example. The short time existence result (theorem (3–6)) then applies to give existence on a slightly longer time interval.

To prove this result, only very crude estimates are required. Choose the origin of $\mathbb{R}^{n+1}$ to be at the centre of a sphere which achieves the inner radius at time t_0, and consider the equation (3–3) which describes the evolution as a graph over a sphere about the origin. Note that the radial length r is in the range $[\rho_0, 2\rho_+(0)]$. Also note that $|\bar{\nabla} r|$ is bounded, since $0<\rho_0\leq\langle\varphi,\nu\rangle=\frac{r^2}{\sqrt{r^2+|\bar{\nabla} r|^2}}$ by the convexity of the hypersurface. It follows that equation (3–3) is uniformly parabolic on this time interval. If f is a convex function of the principal curvatures, then $\mathcal{F}$ is concave as a function of the second derivatives of r; if f is a concave function of the principal curvatures, then let $\sigma=-r$. Then it is clear that the time derivative of σ is given by a concave function of the second derivatives of σ. The regularity estimates now follow from the regularity theory for concave, nonlinear parabolic equations developed by Krylov and others—see in particular **[K]**, section (5–5). The other conditions required to apply these estimates are easily checked in view of the bounds on r and $|\bar{\nabla} r|$ mentioned above. ∎

Theorem 6–2. *The maximal time of existence T of the solution φ_0 is finite, and the solution φ converges uniformly to a constant p in $\mathbb{R}^{n+1}$ as this final time is approached.*

Proof : Suppose $\psi_0 : S^n \to \mathbb{R}^{n+1}$ gives at time $t = 0$ a sphere which encloses the initial hypersurface $\varphi_0(M^n)$. The solution ψ of equation (1–1) with initial condition ψ_0 converges to a constant in finite time given by $\frac{1}{2}\rho_+^2(0)$. It follows that T must be finite, since the images of φ and ψ remain disjoint—to prove this, consider the evolution of the function $|\varphi(x) - \psi(y)|$ for $(x, y) \in M^n \times S^n$: If a minimum of this function occurs at a point (x, y) at time t, it can be seen that $\mathcal{W}(x) \geq \mathcal{W}(y)$, so by the monotonicity condition of (3–1), $F(\mathcal{W}(x)) \geq F(\mathcal{W}(y))$, and the distance is nondecreasing by the maximum principle.

This proves that ρ_- approaches zero as t approaches T, in view of the previous theorem. Note that the regions enclosed by the hypersurfaces $\varphi_t(M^n)$ are decreasing and nonempty on [0,T). Hence they have nontrivial intersection. Let p be any point in this intersection. By theorem (5–1), we have the estimate:

$$\sup_M |\varphi_t - p| \leq \rho_+ \leq C_2 \rho_-$$

and the convergence follows. ∎

7. Convergence to a Sphere

In this chapter a natural rescaling of equation (1–1) is considered. The evolution of a sphere suggests an appropriate scaling factor: Suppose $M^n = S^n$ and $\varphi_0(z) = r_0 z$, so that the initial hypersurface is a round sphere. Then the solution of (1–1) is given by $\varphi_t(z) = \sqrt{2(T-t)}z$, on the interval [0,T) where $T = \frac{r_0^2}{2}$. This agrees with the scaling factor given in theorem (1–2).

Define a new time parameter τ by $\tau = -\frac{1}{2}\ln(1-\frac{t}{T})$, and denote the rescaled immersions by $\tilde{\varphi}_\tau = (2(T-t))^{-\frac{1}{2}}(\varphi_t - p)$, where p is given in theorem (6–1). A direct calculation shows that the following equation is satisfied for τ in the interval $[0,\infty)$:

$$\text{(7–1)} \qquad \frac{\partial}{\partial\tau}\tilde{\varphi}(x,\tau) = -F\left(\tilde{\mathcal{W}}(x,\tau)\right)\tilde{\nu}(x,\tau) + \tilde{\varphi}(x,\tau)$$

where the geometric quantities associated with the rescaled immersions are distinguished by a tilde.

Some uniform estimates for the rescaled equation can be proved immediately:

Lemma 7–2.

$$\frac{1}{C_2} \le \tilde{\rho}_-(\tau) \le \tilde{\rho}_+(\tau) \le C_2.$$

for all $\tau \ge 0$, where C_2 is given by theorem (5–4).

Proof : Recall the definition (2–15) of the inner and outer radii. Let $\xi \in \mathbb{R}^{n+1}$ be such that the sphere $S_{\rho_+(t)}(\xi)$ encloses $\varphi_t(M)$. By the maximum principle (compare the proof of theorem (6–2)), $\varphi_{t'}(M)$ remains enclosed by $S_{\rho(t)}(\xi)$ for all t' in the range (t', T), where $\rho(t) = \sqrt{\rho_+^2(t) - 2(t'-t)}$. Thus we have the inequality $\rho_+^2(t') \leq \rho_+^2(t) - 2(t'-t)$. Since the solution exists up to time τ, we have $\rho_+^2(t) \geq 2(T-t)$. Equivalently, $\tilde{\rho}_+(\tau) \geq 1$. Then theorem (5–4) implies $\tilde{\rho}_-(\tau) \geq \frac{1}{C_2}$.

Now let $\xi' \in \mathbb{R}^{n+1}$ be such that the sphere $S_{\rho_-(t)}(\xi')$ is enclosed by $\varphi_t(M)$. As before, the maximum principle shows that $\rho_-^2(t') \geq \rho_-^2(t) - 2(t'-t)$. By theorem (6–1), we know that $\lim_{t' \to T} \rho_-^2(t') = 0$, so we have $\rho_-(t) \leq \sqrt{2(T-t)}$, and by theorem (5–4), $\tilde{\rho}_+(\tau) \leq C_2$. ∎

Corollary 7–3.

$$\sup \tilde{F} \geq \frac{1}{C_2}.$$

Proof: Consider a point in contact with an enclosing sphere at time τ and use the estimate $\tilde{\rho}_+ \leq C_2$. ∎

The following result is adapted from an estimate due to Tso [**Ts**]:

Theorem 7–4. *There is some constant C_3 such that the following estimate holds for sufficiently large times τ:*

$$\sup \tilde{F}(\tau) \leq C_3.$$

Proof: Consider any time t_0 in the range $(0,T)$, and choose the origin of $\mathbb{R}^{n+1}$ to be at the centre of a sphere of radius $\rho_-(t_0)$ which is enclosed by $\varphi_{t_0}(M)$. On the time interval $[0,t_0]$ we have $\langle \varphi, \nu \rangle \geq \rho_-(t_0)$. Consider the following evolution equation which is derived from (1–1), (3–9), and (3–12):

$$(\frac{\partial}{\partial t}-\mathcal{L})\frac{F}{2\langle\varphi,\nu\rangle-\rho_-(t_0)}=\frac{2\dot{F}g^*}{2\langle\varphi,\nu\rangle-\rho_-(t_0)}\left(\nabla\langle\varphi,\nu\rangle\otimes\nabla\left(\frac{F}{2\langle\varphi,\nu\rangle-\rho_-(t_0)}\right)\right)$$

$$+\frac{F\left(4F-\rho_-(t_0)\dot{F}(\mathcal{W}^2)\right)}{\left(2\langle\varphi,\nu\rangle-\rho_-(t_0)\right)^2}. \tag{7–5}$$

Theorem (4–1) ensures that $\dot{F}(\mathcal{W}^2)\geq CF^2$ for some constant C. Applying the maximum principle, we have an estimate for $Q=\sup_t\left(\frac{F}{2\langle\varphi,\nu\rangle-\rho_-(t_0)}\right)$:

$$\begin{aligned}\frac{d}{dt}Q&\leq Q^2\left(4-C\rho_-(t_0)F\right)\\&\leq Q^2\left(4-C(\rho_-(t_0))^2Q\right).\end{aligned}$$

It follows that $Q(t_0)\leq\max\{Q(0),\frac{4}{C(\rho_-(t_0))^2}\}$. Since $\rho_-(t_0)$ tends to zero, we can choose t_0 large enough to ensure that the second of these terms is the greater, and we have $\sup\tilde{F}(\tau)\leq C_3$, where $C_3=\frac{16C_2^2}{C}$. ∎

Lemma 7–6. *There are constants $C_4>0$ and $\tau_0>0$ such that*

$$\inf\tilde{F}\geq C_4$$

for all $\tau>\tau_0$.

Proof: Let $\tau_1\geq 0$. Choose the origin of $\mathbb{R}^{n+1}$ to be at the centre of the largest enclosed circle at time τ_1, and consider the rescaled equation parametrised as a spherical graph. At time τ_1 we have $\frac{1}{C_2}\leq\tilde{\rho}_-(\tau_1)\leq\tilde{r}\leq 2\tilde{\rho}_+(\tau_1)\leq 2C_2$. A comparison with the evolution of enclosed and enclosing spheres ensures that we also have $\frac{1}{2C_2}\leq\tilde{\rho}_-(\tau_1)\leq\tilde{r}\leq 2\tilde{\rho}_+(\tau_1)\leq 2C_2$ on the entire interval $[\tau_1,\tau_1+\Delta]$, where $\Delta=\frac{1}{2}\ln\left(\frac{4C_2^2-1}{4C_2^2-4}\right)>0$ since $C_2>1$. By convexity we also have $|\bar{\nabla}\tilde{r}|\leq 8C_2^3$, since $\langle\tilde{\varphi},\tilde{\nu}\rangle\geq\tilde{\rho}_-$. Consider the evolution equation for $\tilde{F}$, given by (3–16):

$$(\frac{\partial}{\partial\tau}-\tilde{\mathcal{L}})\tilde{F}=\dot{F}(\tilde{\mathcal{W}}^2)-\tilde{F}-\frac{1}{r}\dot{F}\tilde{g}^*\left(\bar{\nabla}\tilde{F}\otimes\bar{\nabla}\tilde{r}+\bar{\nabla}\tilde{r}\otimes\bar{\nabla}\tilde{F}\right). \tag{7–7}$$

Note that this can be written in a simple form:

$$(\frac{\partial}{\partial \tau} - \bar{\mathcal{L}})\tilde{F} = A.\bar{\nabla}\tilde{F} + B\tilde{F}$$

where $\bar{\mathcal{L}}$ is a uniformly elliptic operator, and A and B are bounded. Corollary (7–3) and the Harnack inequality (see [**K**], section (4.2)) give the existence of a constant C_4 such that $\inf_{\tau_1+\Delta} \tilde{F} \geq C_4$. Since τ_1 is arbitrary, this gives the result for $\tau \geq \Delta$. ∎

The last two results, combined with the result of theorem (4–1), show that the principal curvatures remain in a compactly contained subregion of Γ_+ throughout the evolution. The results of [**K**] can now be applied to give the following regularity result—note that boundedness in C^0 follows immediately from (7–2) since the final point is always enclosed:

Lemma 7–8. *The support function $\tilde{s} : S^n \to \mathbb{R}$ of the rescaled immersions is uniformly bounded in C^k for all positive integers k.*

Note that this does not immediately imply uniform C^k bounds for the immersions $\tilde{\varphi}$, since only the geometric properties (and not the parametrisation) have been controlled. The result gives convergence of $\tilde{s}$ in C^∞ for a subsequence of times $\{\tau_k\}$ to a support function s_∞ of a compact, strictly convex hypersurface. In fact this is already enough to deduce the full result of theorem (1–2): It is easy to show that the limit s_∞ is itself part of a solution to the original equation (1–1). The proof of theorem (4–1) gives a scaling-invariant quantity which decreases strictly unless it is everywhere constant. Hence this quantity must be constant on the limit, which implies that s_∞ is the support function of a sphere.

I will give a different proof here, because it highlights several useful properties of the evolution equations. In particular, I will derive a simple Poincaré inequality, and investigate the evolution equations satisfied by the derivatives of the curvature. The following estimate is used to show that this hypersurface must be the unit sphere:

Lemma 7–9. *If f is convex, let $\eta = \frac{\tilde{F}}{\tilde{H}_{-1}}$, where H_{-1} is the harmonic mean curvature, given by $\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i^{-1}\right)^{-1}$. If f is concave, let $\eta = \frac{|\tilde{\mathcal{W}}|}{\tilde{F}}$. Then for some sufficiently large q and some constant C_5, the following estimate holds:*

$$\frac{\partial}{\partial\tau}\int_{M^n}\tilde{K}\left(\eta^q - \eta_0^q\right)d\tilde{\mu} \le -C_5\int_{M^n}\tilde{K}\frac{|\tilde{\nabla}\tilde{\mathcal{W}}|^2}{|\tilde{\mathcal{W}}|^2}d\tilde{\mu} \tag{7–10}$$

where $\eta_0 = \eta(1,\dots,1)$.

Remark : The functions $\tilde{H}_{-1}$ and $|\tilde{\mathcal{W}}|$ can be replaced with other homogeneous degree one functions, as long as they are strictly concave and convex respectively except in radial directions in Γ_+.

Proof: First calculate the evolution equations for η—From the equation (3–13), any function Φ which is homogeneous of degree one in $\tilde{\mathcal{W}}$ evolves as follows:

$$\left(\frac{\partial}{\partial\tau} - \tilde{\mathcal{L}}\right)\Phi = (\dot{\Phi}\ddot{F} - \dot{F}\ddot{\Phi})(\tilde{\nabla}\tilde{\mathcal{W}}, \tilde{\nabla}\tilde{\mathcal{W}}) + \Phi\dot{F}(\tilde{\mathcal{W}}^2) - \Phi. \tag{7–11}$$

In order to make use of the term involving the second derivatives of F and Φ, we use the following result:

Lemma 7–12. *Suppose $\Phi = \phi(\lambda)$ satisfies conditions (3–1), and ϕ is strictly convex (concave) in non-radial directions in Γ_+ (Equivalently,* Hessϕ *has only one zero eigenvalue at each point in Γ_+). Then there is a constant C such that*

$$\begin{aligned} g^*\ddot{\Phi}(\nabla\mathcal{W}, \nabla\mathcal{W}) &\ge C\frac{|\nabla\mathcal{W}|^2}{|\mathcal{W}|} && \text{if } \phi \text{ is convex;} \\ &\le -C\frac{|\nabla\mathcal{W}|^2}{|\mathcal{W}|} && \text{if } \phi \text{ is concave.} \end{aligned} \tag{7–13}$$

Proof : For ϕ strictly convex in non-radial directions, we have for any ξ in $\mathbb{R}^n$ the inequality:

$$\frac{\partial^2 \phi}{\partial \lambda_i \partial \lambda_j} \xi^i \xi^j \geq C \sum_{i,j} \frac{|\lambda_i \xi^j - \lambda_j \xi^i|^2}{|\lambda|^3}. \tag{7–14}$$

Now we use the expression (2–24), working in local coordinates which diagonalise $\mathcal{W}$:

$$g^* \ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) = \sum_{k,i,j} \left(\frac{\partial^2 \phi}{\partial \lambda_i \partial \lambda_j} (\nabla_k \mathcal{W}_i^i)(\nabla_k \mathcal{W}_j^j) \right) + \sum_{k, i \neq j} \frac{\frac{\partial \phi}{\partial \lambda_i} - \frac{\partial \phi}{\partial \lambda_j}}{\lambda_i - \lambda_j} (\nabla_k \mathcal{W}_i^j)^2$$

Now apply (7–14) to the expression (2–22) to show that

$$\frac{\frac{\partial \phi}{\partial \lambda_i} - \frac{\partial \phi}{\partial \lambda_j}}{\lambda_i - \lambda_j} \geq \frac{C}{|\lambda|}$$

so we have

$$g^* \ddot{\Phi}(\nabla \mathcal{W}, \nabla \mathcal{W}) \geq \frac{C}{|\mathcal{W}|^3} \sum_{i,j,k} |\lambda_i \nabla_k \mathcal{W}_j^j - \lambda_j \nabla_k \mathcal{W}_i^i|^2 + \frac{C}{|\mathcal{W}|} \sum_{k, i \neq j} |\nabla_k \mathcal{W}_i^j|^2.$$

Consider a point $(\lambda, B) \in E \subset \Gamma_+ \times T^*M \otimes T^*M \otimes T^*M$, where E is the compact set $\{\lambda \in \Gamma_+, B \text{ totally symmetric}, |\lambda| = |B| = 1\}$. Then if $g^* \ddot{\Phi}(B, B) = 0$, we have $B_{kij} = 0$ for $i \neq j$, and $\frac{B_{kjj}}{\lambda_j} = \frac{B_{kii}}{\lambda_i}$ for every i, j, k. But since B is totally symmetric, this implies $B = 0$. Hence $g^* \ddot{\Phi}(B, B)$ has a strictly positive lower bound C on the compact set E, and extends by homogeneity to general λ and B:

$$g^* \ddot{\Phi}(B, B) \geq \frac{C|B|^2}{|\lambda|}$$

The result follows since the Codazzi equations (2–8) guarantee that $\nabla \mathcal{W}$ is totally symmetric.

If Φ is concave, the result follows by considering $-\Phi$, which is convex. ∎

Proof of Lemma (7–9), **contd.** : If F is convex, the evolution equation for η is given as follows:

$$(\frac{\partial}{\partial\tau}-\tilde{\mathcal{L}})\eta=\frac{2}{\tilde{H}_{-1}}\dot{F}\tilde{g}^*\left(\tilde{\nabla}\tilde{H}_{-1}\otimes\tilde{\nabla}\eta\right)+\eta\frac{(\dot{F}\ddot{H}_{-1}-\dot{H}_{-1}\ddot{F})}{\tilde{H}_{-1}}(\tilde{\nabla}\tilde{\mathcal{W}},\tilde{\nabla}\tilde{\mathcal{W}}).$$

If F is concave, we have the similar equation:

$$(\frac{\partial}{\partial\tau}-\tilde{\mathcal{L}})\eta=\frac{2}{\tilde{F}}\dot{F}\tilde{g}^*\left(\tilde{\nabla}\tilde{F}\otimes\tilde{\nabla}\eta\right)+\frac{(\dot{F}|\ddot{I\!I}|-|\dot{I\!I}|\ddot{F})}{\tilde{F}}(\tilde{\nabla}\tilde{\mathcal{W}},\tilde{\nabla}\tilde{\mathcal{W}}).$$

We proceed similarly in both cases—the term involving $\tilde{\nabla}\eta$ is estimated using the Schwarz inequality and theorem (4–1), and the last term is estimated by Lemma (7–12):

$$(\frac{\partial}{\partial\tau}-\tilde{\mathcal{L}})\eta\leq C\left(\frac{\epsilon|\tilde{\nabla}\tilde{\mathcal{W}}|^2}{|\tilde{\mathcal{W}}|^2}+\frac{|\tilde{\nabla}\eta|^2}{\epsilon}\right)-C\frac{|\tilde{\nabla}\tilde{\mathcal{W}}|^2}{|\tilde{\mathcal{W}}|^2}$$

for every $\epsilon>0$. Integrating and using (4–1) again, we find:

$$\begin{aligned}\frac{d}{d\tau}\int_M\tilde{K}\eta^q d\tilde{\mu}\leq -Cq^2\int_M\tilde{K}\eta^{q-2}|\tilde{\nabla}\eta|^2d\tilde{\mu}-Cq\int_M\tilde{K}\eta^{q-1}\frac{|\tilde{\nabla}\tilde{\mathcal{W}}|^2}{|\tilde{\mathcal{W}}|^2}d\tilde{\mu}\\+Cq\int_M\tilde{K}\eta^{q-1}\left(\epsilon^{-1}|\tilde{\nabla}\eta|^2+\epsilon\frac{|\tilde{\nabla}\tilde{\mathcal{W}}|^2}{|\tilde{\mathcal{W}}|^2}\right)d\tilde{\mu}.\end{aligned}$$

Note that the derivative of the term $\tilde{K}d\tilde{\mu}$ introduces only gradients of curvature, and no curvature terms. This is due to its interpretation as the area element on S^n under the Gauss map. This gives the result by choosing q large enough and choosing ϵ appropriately. ∎

Corollary 7–15. $s_\infty=1$.

Proof : The previous theorem gives a strictly decreasing quantity if $\tilde{\nabla}\tilde{\mathcal{W}}$ is not everywhere zero. It follows that s_∞ is the support function of a sphere, since this is the only compact convex hypersurface with constant curvature. From the proof of (7–2) we have $\rho_-\leq 1$ and $\rho_+\geq 1$, which implies that the sphere must have radius 1. Suppose the centre of the sphere is at some point $p' \neq 0$. Then for some sufficiently large time τ_k, $\tilde{\varphi}_{\tau_k}(M)$ is contained in a ball of radius $1 + \frac{3}{4}|p'|^2$ about p'. By the maximum principle, $\tilde{\varphi}_\tau(M)$ is contained in a ball of radius $e^{\tau-\tau_k}\sqrt{\frac{3}{4}|p'|^2 + e^{2(\tau_k-\tau)}}$ about the point $e^{\tau-\tau_k}p'$. For $\tau > \tau_k + \ln\frac{2}{|p'|}$ this ball does not contain the origin, and therefore the unrescaled immersion does not contain the point p, which contradicts theorem (6–1). Thus the limit sphere must have centre at the origin, and $s_\infty \equiv 1$. ∎

Theorem 7–16. *$\tilde{s}_\tau$ converges to s_∞ as τ approaches ∞, exponentially in C^∞. That is, there is a positive number δ, and constants $\hat{C}_k$ for every k in such that*

$$|\tilde{\nabla}^{(k)}(\tilde{s} - s_\infty)|^2 \leq \hat{C}_k e^{-\delta\tau}.$$

Proof: Simons's Identity (3–15) can be used to prove a simple inequality which allows us to use the good gradient term in equation (7–11):

$$\begin{aligned}\left(g^*\mathrm{Hess}_{\tilde{\nabla}}\tilde{\mathcal{W}}\right)(\tilde{\mathcal{W}} \otimes \mathrm{Id}) = \left(g^*\mathrm{Hess}_{\tilde{\nabla}}\tilde{\mathcal{W}}\right)(\mathrm{Id} \otimes \tilde{\mathcal{W}}) + \tilde{\mathcal{W}}^\dagger(Id)(\tilde{\mathcal{W}}^\dagger)^2(\tilde{\mathcal{W}}) \\ - \tilde{\mathcal{W}}^\dagger(\tilde{\mathcal{W}})(\tilde{\mathcal{W}}^\dagger)^2(\mathrm{Id}).\end{aligned}$$

Integrating over M^n and using integration by parts gives:

$$\int_{M^n} Z\,d\tilde{\mu} = \int_{M^n}\left(n^2|\tilde{\nabla}\tilde{H}|^2 - |\tilde{\nabla}\tilde{\mathcal{W}}|^2\right)d\tilde{\mu}$$

where $\tilde{H}$ is the mean curvature, given by $\frac{1}{n}\sum_{i=1}^n \tilde{\lambda}_i$, and Z is given by the expression $\sum_{i,j}\tilde{\lambda}_i\tilde{\lambda}_j\left(\tilde{\lambda}_i - \tilde{\lambda}_j\right)^2$. In view of theorem (4–1) and the definition of η, we have an estimate $\eta^q - \eta_0^q \leq CZ$. The integral above now gives:

$$\int_{M^n}\tilde{K}\,(\eta^q - \eta_0^q)\,d\tilde{\mu} \leq C\int_{M^n}|\tilde{\nabla}\tilde{\mathcal{W}}|^2 d\tilde{\mu} \tag{7–17}$$

where q is given in lemma (7–9). If we define $\sigma = \eta^q - \eta_0^q$, then $\int_M \sigma\,d\tilde{\mu}$ is exponentially decreasing, by (7–17) and (7–10). Since $\tilde{\nabla}\sigma$ is uniformly bounded, this also proves that $\sup_M \sigma$ is exponentially decreasing with some exponent δ. Note that we can assume that q is large enough to give the following estimate for the evolution of σ:

$$(\frac{\partial}{\partial\tau} - \tilde{\mathcal{L}})\sigma \leq -C_0'|\tilde{\nabla}\tilde{\mathcal{W}}|^2. \tag{7–18}$$

Now consider the following evolution equation, which can be calculated by differentiating (3–13), and using the derivative commutation formula (2–5), the Gauss equations (2–9), and the results of theorem (4–1):

$$(\frac{\partial}{\partial\tau} - \tilde{\mathcal{L}})|\tilde{\nabla}\tilde{\mathcal{W}}|^2 \leq -C_1'|\tilde{\nabla}^{(2)}\tilde{\mathcal{W}}|^2 + C_1''|\tilde{\nabla}\tilde{\mathcal{W}}|^2 + C_1'''|\tilde{\nabla}\tilde{\mathcal{W}}|^4. \tag{7–19}$$

More generally, we have the following expression for the evolution of the higher derivatives of the curvature:

$$\begin{aligned}(\frac{\partial}{\partial\tau} - \tilde{\mathcal{L}})|\tilde{\nabla}^{(k)}\tilde{\mathcal{W}}|^2 = &- 2\dot{F}\tilde{g}^*\left(\tilde{\nabla}^{(k+1)}\tilde{\mathcal{W}}, \tilde{\nabla}^{(k+1)}\tilde{\mathcal{W}}\right)\\ &+ \tilde{\nabla}^{(k)}\tilde{\mathcal{W}} * \sum_{\mathcal{I}_{k+1}} \tilde{\nabla}^{(i_1+1)}\tilde{\mathcal{W}} * \tilde{\nabla}^{(i_2)}\tilde{\mathcal{W}} * \cdots * \tilde{\nabla}^{(i_\ell)}\tilde{\mathcal{W}} * \mathcal{C}(\tilde{\mathcal{W}})\\ &+ \tilde{\nabla}^{(k)}\tilde{\mathcal{W}} * \sum_{\mathcal{I}_k} \tilde{\nabla}^{(i_1)}\tilde{\mathcal{W}} * \cdots * \tilde{\nabla}^{(i_\ell)}\tilde{\mathcal{W}} * \mathcal{C}(\tilde{\mathcal{W}}).\end{aligned}$$

where $A * B$ denotes any linear combination of tensors formed by contractions of A on B by the metric $\tilde{g}$, and $\mathcal{C}(\tilde{\mathcal{W}})$ denotes any smooth function of $\tilde{\mathcal{W}}$. For any positive integer m, $\mathcal{I}_m$ is the set of sequences of positive integers $\{i_1, \ldots . i_\ell\}$ such that $i_1 + i_2 + \cdots + i_\ell = m$. This gives the estimate:

$$(\frac{\partial}{\partial\tau} - \tilde{\mathcal{L}})|\tilde{\nabla}^{(k)}\tilde{\mathcal{W}}|^2 \leq -C_k'|\tilde{\nabla}^{(k+1)}\tilde{\mathcal{W}}|^2 + C_k''|\tilde{\nabla}^{(k)}\tilde{\mathcal{W}}|^2 + C_k'''\sum_{j<k}|\tilde{\nabla}^{(j)}\tilde{\mathcal{W}}|^2. \tag{7–20}$$

Combining equations (7–18) and (7–19), we obtain the following:

$$(\frac{\partial}{\partial\tau} - \tilde{\mathcal{L}})(|\tilde{\nabla}\tilde{\mathcal{W}}|^2 + \frac{C_1'' + 3\delta}{C_0'}\sigma) \leq -3\delta|\tilde{\nabla}\tilde{\mathcal{W}}|^2 + C_1'''|\tilde{\nabla}\tilde{\mathcal{W}}|^4. \tag{7–21}$$

Since we have convergence to a sphere on a subsequence of times, we can choose a time τ_k such that $|\tilde{\nabla}\tilde{\mathcal{W}}|^2 + \frac{C_1''+3\delta}{C_0'}\sigma < \frac{\delta}{C_1'''}$. Equation (7–21) shows that this is preserved for all $\tau > \tau_k$, and consequently the following estimate holds for these times, if we define $Q_1 = |\tilde{\nabla}\tilde{\mathcal{W}}|^2 + \frac{C_1''+3\delta}{C_0'}\sigma$:

$$(\frac{\partial}{\partial\tau} - \tilde{\mathcal{L}})Q_1 \leq -2\delta Q_1 + 2\delta\frac{C_1''+3\delta}{C_0'}\sigma.$$

This can be integrated using the estimate $\sigma < Ce^{-\delta\tau}$ to give:

$$Q_1(\tau) < Ce^{-\delta\tau}.$$

This is already sufficient to show convergence, or even exponential convergence in C^k for any k, using standard interpolation inequalities (see [**GT**], section (6.8)). To obtain exponential convergence with the same exponent δ, one can use equation (7–20)—to control the kth derivatives of $\tilde{\mathcal{W}}$, a sufficient quantity of lower derivatives are added to give a good evolution equation. ∎

Theorem 7–22. *$\tilde{\varphi}_\tau$ converges exponentially with exponent δ in C^∞ to a smooth immersion φ_∞ with image equal to the unit sphere.*

Proof : Consider the rescaled evolution equation for the metric $\tilde{g}$:

$$\frac{\partial}{\partial\tau}\tilde{g}(u,u) = -2\tilde{F}\tilde{I\!I}(u,u) + 2\tilde{g}(u,u).$$

Note that $\tilde{F}$ converges exponentially to 1, and $\frac{\tilde{I\!I}(u,u)}{\tilde{g}(u,u)}$ converges exponentially to the value 1. Hence we have the following estimate:

$$\left|\frac{\partial}{\partial\tau}\ln(\tilde{g}(u,u))\right| \leq Ce^{-\delta\tau}.$$

This has finite integral, which gives uniform bounds above and below for the length under $\tilde{g}$ of any nonzero vector u in TM. This shows that the map $\tilde{\varphi}$ does not degenerate as we take the limit $\tau \to \infty$; higher regularity is proved similarly, using the integrability of the decaying exponential bounds obtained above. ∎

Remark : In the case of the mean curvature flow, the analysis is particularly simple: The results of [**K**] can be replaced by an explicit Harnack inequality [**Ha3**] and interior estimates [**EH**]. Many other flows also have explicit Harnack inequalities as proved in section II of this thesis.

Section II

HARNACK INEQUALITIES

FOR

EVOLVING HYPERSURFACES

1. Introduction

Harnack inequalities for parabolic equations originate with the work of Moser [**Mo**] who treated the case of linear divergence-form equations. In this context the inequality estimates a solution from below in terms of the largest value it attains on an earlier region of the parabolic domain. Inequalities of this type have recently appeared for many geometric evolution equations, including several quasilinear and fully nonlinear examples. These new developments began with Li and Yau [**LY**], who showed how to obtain a Harnack inequality for the heat equation by clever use of the parabolic maximum principle. Similar techniques were employed by Hamilton, who proved Harnack inequalities for various nonlinear evolution equations—the flow of Riemannian metrics by their Ricci curvature in two dimensions [**Ha2**], the mean curvature flow of hypersurfaces in Euclidean space, and several scalar equations [**Ha3**]. Chow has treated flows of hypersurfaces in Euclidean space by powers of the Gauss curvature [**Ch3**], and also the flow of Riemannian metrics by the gradient of the Yamabe functional [**Ch4**]. Most recently, Hamilton has proved a Harnack inequality for the higher-dimensional Ricci curvature flow [**Ha5**].

In this section I will prove Harnack inequalities for a wide class of fully nonlinear evolution equations for hypersurfaces. The first step is to show that a certain natural quantity (to leading order, the time derivative of the speed of the hypersurface) satisfies a simple and useful evolution equation (Lemma (5–1)). Hamilton and Chow accomplish this step by performing a long and cumbersome calculation with an astonishingly simple result. Here the calculation is made transparent by a natural reparametrisation of the flow equations, discussed in chapter 3. The parabolic maximum principle can be applied to this evolution equation to deduce a differential inequality for the speed (Theorems (5–6), (5–11)). In many cases this in turn can be integrated to give a Harnack inequality for the speed. This integration can be performed in various ways, two of which are described here: Theorem (5–18) gives an estimate which imitates the methods of previous work, and theorem (5–22) describes an alternative estimate which seems more useful in many cases. An integral estimate (an *entropy inequality*) can also be obtained for certain flows by integrating the differential equation of lemma (5–1) over the whole manifold instead of using the parabolic maximum principle (Theorem (5–27)). The results apply for a wide range of flows of hypersurfaces, including the mean curvature flow, the Gauss curvature flows, and many other examples. A large class of new flows is considered, and some examples are discussed in chapter 4.

As in section I, we consider a compact, smooth n-dimensional manifold M^n, and a smooth family of immersions $\varphi : [0, T) \times M^n \to \mathbb{R}^{n+1}$ of M^n into Euclidean space $\mathbb{R}^{n+1}$, satisfying an equation of the following form:

$$\frac{\partial}{\partial t}\varphi(x,t) = S(\nu(x,t),t))\nu(x,t) \tag{1–1}$$

where $\nu : M^n \times [0,T) \to S^n$ is a unit normal to the hypersurface $\varphi_t(M^n)$ at the point $\varphi_t(x)$. The function $S_t : S^n \to \mathbb{R}$ gives the speed of motion of the hypersurfaces through $\mathbb{R}^{n+1}$. This must be of a special form, since we wish to ensure that equation (1–1) is a parabolic, second order system of partial differential equations, which is invariant under diffeomorphisms of M^n, translations of φ in $\mathbb{R}^{n+1}$, and translations in time. This allows considerably more general equations than the class considered in section I; the precise form implied by these conditions will be discussed in chapter 3. In many cases we may wish to consider the stronger condition that equation (1–1) must be invariant under all isometries of $\mathbb{R}^{n+1}$. Equations which satisfy this are called *isotropic*, while those that only satisfy the weaker condition are called *anisotropic*. The equations of part I are all isotropic.

The class of equations considered includes all of the hypersurface flows which have been considered previously: The Gauss curvature flows considered by Tso [**Ts**] and Chow [**Ch1**]; the expanding hypersurface flows considered by Urbas ([**U1**],[**U2**]), Huisken ([**Hu4**]), and Gerhardt ([**Ge**]); and some anisotropic flows of the form $S = -\mu(\nu)H$, considered as simple models of crystal growth by Cahn, Handwerker and Taylor [**CHT**]. Further important examples are given in section IV of this thesis, where anisotropic flows are central to a new proof of the Aleksandrov-Fenchel inequalities for convex bodies.

2. Notation and Conventions

Much of the notation for this section is the same as in part I. The most important new notation concerns the parametrisation of hypersurfaces by the Gauss map. This idea, introduced in part I, will be developed more fully here.

As noted in chapter 2 of part I, the Gauss map is a diffeomorphism for a strictly convex compact hypersurface. It follows that we can use the Gauss map to parametrise such hypersurfaces. This gives an immersion $\varphi : S^n \to \mathbb{R}^{n+1}$, for which the Gauss map ν is the identity map on S^n. The support function $s : S^n \to \mathbb{R}$ gives the perpendicular distance from the origin of the tangent plane at $\varphi(z)$. Hence the immersion φ must have this form:

$$\varphi(z) = s(z)z + a(z) \tag{2–1}$$

where $a(z)$ is a vector tangent to S^n at z, for each z. Differentiating this expression in a tangential direction u gives the following result:

$$\begin{aligned} T\varphi(u) &= (D_u s)z + su + D_u a \\ &= (D_u s)z + su + \bar{\nabla}_u a - \bar{g}(u, a)z \end{aligned} \tag{2–2}$$

where we have applied to the sphere S^n the definitions (I.2–6) and (I.2–7) of the connection and the second fundamental form. The vector a can be deduced from the fact that the tangent space $T_z\varphi(T_zS^n)$ is parallel to the tangent space T_zS^n— this implies that the component of the expression (2–2) in direction z must be zero. Hence we have $\bar{g}(u, a) = D_u s$ for every tangential vector u, and so $a(z) = \bar{\nabla}s$, the gradient of s with respect to the metric $\bar{g}$ on the sphere. The immersion is therefore given by the following expression

$$\varphi(z) = s(z)z + \bar{\nabla}s. \tag{2–3}$$

An expression for the curvature also follows from the calculation above: Recall that the Weingarten curvature $\mathcal{W}$ is given by the expression $\mathcal{W}(u) = T\varphi^{-1} \circ T\nu(u)$, from equation (I.2–10). In the present situation we have $T\nu = \mathrm{Id}$, and so $\mathcal{W}^{-1} = T\varphi$. The calculation (2–2) and the expression for the vector a therefore give:

$$\begin{aligned} \mathcal{W}^{-1}(u) &= \bar{\nabla}_u(\bar{\nabla} s) + s\mathrm{Id}(u) \\ &= \left(\bar{g}^* \mathrm{Hess}_{\bar{\nabla}} s + s\mathrm{Id}\right)(u). \end{aligned} \tag{2–4}$$

For convenience we will denote the map $\bar{g}^*\mathrm{Hess}_{\bar{\nabla}} s + \mathrm{Id}s$ by A.

It is useful to have expressions for the metric and connection of the hypersurface in terms of s and A:

$$g(u, v) = \bar{g}(A(u), A(v)) \tag{2–6}$$

$$\nabla_u v = \bar{\nabla}_u v + A^{-1}\left(\bar{\nabla} A(u, v)\right) \tag{2–7}$$

Another useful equation, which can be deduced directly from the form of the map A, is a form of the Codazzi equations:

$$\bar{\nabla} A(u, v) = \bar{\nabla} A(v, u). \tag{2–8}$$

The great advantage of the support function is that it allows us to consider a family of convex hypersurfaces simply as an evolving scalar function defined on the sphere. This makes things much simpler than the more abstract framework allowing arbitrary parametrisations, since we no longer have different descriptions of the same hypersurface. Furthermore, the identification with the sphere provides a time-independent metric and connection, which vastly simplifies many calculations—including especially those presented here for the proof of the Harnack inequalities.

The expression (2–4) allows us to use the support function to calculate functions of the curvature of a hypersurface. For example, a function $F(\mathcal{W})$ (such as the speed functions of part I) gives rise to a 'dual' function $\Phi(\bar{g}^*\mathrm{Hess}_{\bar{\nabla}}s + \mathrm{Id}s)$, defined according to the following equation:

$$\Phi(A) = -F(A^{-1}) \tag{2–9}$$

for every positive definite map A. The application of these ideas to the evolution equations will be developed fully in the next chapter.

Examples : The mean curvature H is given by the trace of the Weingarten map, which is the trace of the inverse of the map A. The harmonic mean curvature is the inverse of the trace of the inverse of the Weingarten map, or the reciprocal of the trace of A. The Gauss curvature is the determinant of $\mathcal{W}$, or the inverse of the determinant of A. More examples are given in chapter 4.

3. The Evolution Equations

In this section we will work with the support function, rather than working explicitly with the hypersurfaces. This contrasts with section I, where we considered the speed as a function of the principal curvatures, and worked with the metric and measure on the hypersufaces. Here we write everything in terms of the support function and the map A defined by (2–4). An important step in this approach is to rewrite equation (1–1) as an evolution equation for the support function.

Theorem 3–1. *Suppose $\varphi : M^n \times [0,T) \to \mathbb{R}^{n+1}$ is a family of strictly convex immersions satisfying* (1–1). *Then the support functions $s : S^n \times [0,T) \to \mathbb{R}$ satisfy the following equation:*

$$\frac{\partial}{\partial t}s(z,t) = S(z,t). \tag{3–2}$$

We require the function S to have the following form:

$$S(z,t) = -F(\mathcal{W}, z) \tag{3–3}$$

for all z in S^n, where $\mathcal{W}$ is the Weingarten map, given by equation (I.2–10). It will be more convenient for us to write this in terms of the map A:

$$S(z,t) = \Phi(A,z). \tag{3–4}$$

Φ is a real function defined on a domain Ω contained in $T^*S^n \otimes TS^n$, the space of linear maps of TS^n. Since we require equation (1–1) to be parabolic, Φ must satisfy a strict monotonicity condition: The derivative $\dot{\Phi}$ of Φ is in $TS^n \otimes T^*S^n$, and is defined for each point z in S^n and each map $\mathcal{Z}$ in Ω by its action on elements

$\mathcal{B}$ of the tensor bundle $T^*S^n \otimes TS^n$:

$$\dot{\Phi}(\mathcal{B}) = \frac{d}{dr}\Phi(\mathcal{Z} + r\mathcal{B}, z)\Big|_{r=0} \tag{3–5}$$

We require that this map be positive definite at each z in S^n and each $\mathcal{Z}$ in Ω. Note that the function F from (3–3) is related to Φ by (2–9). The domain of definition of F is the set Ω' of maps which are inverse to maps in Ω. Note that the derivative $\dot{F}$ of F is positive definite whenever $\dot{\Phi}$ is. Thus our characterisation of parabolicity is the same as in (I.3–1).

Since we usually consider convex hypersurfaces, we will often take Ω to be the set of symmetric positive definite maps of TM. In some circumstances, however, other choices of domain are interesting—see for example the flows used in section VI.

If the equation is isotropic, then S is restricted further: It is given by a symmetric function of the principal curvatures $\lambda_1, \ldots, \lambda_n$. This means also that S is given by a symmetric function of the eigenvalues of the map $A = \bar{g}^*\mathrm{Hess}_{\bar{\nabla}}s + \mathrm{Id}s$, which are called the *principal radii of curvature*.

Examples : The flow by harmonic mean curvature has $S = -H_{-1}$ where $H_{-1} = \left(\frac{1}{n}\sum \lambda_i^{-1}\right)^{-1}$. The flow equation (1–1), written in terms of the principal curvatures, is rather complicated. The coefficients of the elliptic operator $\mathcal{L}$ associated with this flow (introduced in part I) are given by $H_{-1}^2\mathcal{W}^{-2}$. In terms of the support function, however, this flow is much simpler: The equation (3–1) becomes in this case:

$$\frac{\partial}{\partial t}s = -(\bar{\Delta}s + ns)^{-1}$$

where $\bar{\Delta}$ is the Laplacian on S^n. Even simpler is the outward flow by the inverse of the harmonic mean curvature, which has the form:

$$\frac{\partial}{\partial t}s = \bar{\Delta}s + ns.$$

The equation (1–1) extends the parametrisation φ_0 of the initial hypersurface to later hypersurfaces by identifying points on trajectories normal to the hypersurfaces. This will be referred to as the *standard parametrisation* of the flow. The approach adopted here is somewhat different—we identify points which have the same normal direction. This will be referred to as the *Gauss map parametrisation* of the flow.

As in part I, we can find the induced evolution equations for interesting geometric quantities. In the Gauss map parametrisation the details are slightly different from the standard parametrisation—compare the calculations in theorem (I.3–7).

Theorem 3–6. *The following evolution equations hold under the Gauss map parametrisation of the flow* (1–1)*:*

(3–7) $$\frac{\partial}{\partial t}(\mathrm{Hess}_{\bar\nabla} s + \bar g s) = \mathrm{Hess}_{\bar\nabla} S + S\bar g$$

(3–8) $$\frac{\partial}{\partial t} A = \bar g^* \left(\mathrm{Hess}_{\bar\nabla} S\right) + \mathrm{Id} S$$

(3–9) $$\frac{\partial}{\partial t} S = \bar{\mathcal{L}} S + \dot\Phi(\mathrm{Id}) S.$$

where $\bar{\mathcal{L}}$ is the elliptic operator $\dot\Phi \bar g^* \mathrm{Hess}_{\bar\nabla}$.

Proof: The first equation follows simply by differentiating the equation (3–2), since the metric $\bar g$ and connection $\bar\nabla$ are independent of time. The second follows immediately from this. Since Φ depends only on A and z, where z is independent of time, we have:

$$\frac{\partial}{\partial t} S = \dot\Phi\left(\frac{\partial}{\partial t} A\right)$$

which implies equation (3–9). ∎

Example : For the harmonic mean curvature flow we have

$$\frac{\partial}{\partial t} S = S^2(\bar{\Delta} S + nS)$$

which is related to certain porous medium equations. The associated elliptic operator for this flow is $\bar{\mathcal{L}} = S^2 \bar{\Delta}$.

4. Examples

In this chapter I will give some examples of functions $S = \Phi(A)$ which may serve as speeds in the equation (1–1), together with the dual functions F which give the speed in terms of the Weingarten map. I will also note some convexity properties which will be relevant to the results proved later.

4.1 Isotropic Flows : For isotropic flows the speed takes the general form $S = -f(\lambda) = \phi(\kappa)$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ are the principal curvatures, and $\kappa = (\kappa_1, \ldots, \kappa_n)$ are the principal radii of curvature. ϕ is a symmetric function defined on a symmetric domain $\mathcal{C}$ in $\mathbb{R}^n$, and f is the dual function defined by $f(\lambda_1, \ldots, \lambda_n) = -\phi(\lambda_1^{-1}, \ldots, \lambda_n^{-1})$. The domain of definition Ω of the function Φ is given by the set

$$\Omega = \{\mathcal{Z} \in T^*S^n \otimes TS^n : \kappa(\mathcal{Z}) \in \mathcal{C}\}.$$

For flows of convex hypersurfaces we usually choose $\mathcal{C} = \Gamma_+ \subset \mathbb{R}^n$.

Homogeneous examples : It is often natural to consider flows which are invariant under dilations of space, in the sense that a solution remains a solution under this operation, up to a possible rescaling of time. This criterion leads us to consider speed functions which are homogeneous of some degree in the principal curvatures (or the principal radii of curvature). The flows of section I are all isotropic flows with speeds homogeneous of degree one in the principal curvatures (degree -1 in the principal radii of curvature).

The *elementary symmetric functions* are defined by:

$$e^{[k]}(x) = \frac{1}{\binom{n}{k}} \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k} \qquad \text{for } k = 1, \ldots, n.$$

This gives the mean curvature if $f = e_1$, and the Gauss curvature if $f = e_n$. From these a wider class can be defined by taking powers and ratios:

$$e^{[k,l,\alpha]}(x) = \operatorname{sgn}\alpha \cdot \left(\frac{e^{[k]}(x)}{e^{[l]}(x)}\right)^{\alpha} \qquad \text{for } 0 \le l < k \le n \text{ and } \alpha \in \mathbb{R}\backslash\{0\}.$$

(Here $e^{[0]}(x) = 1$). If $f = e^{[k,l,\alpha]}$, then $\phi = e^{[n-l,n-k,-\alpha]}$. The following result is useful:

Lemma. *The functions* $e^{[k,l,\frac{1}{k-l}]}$, *for* $0 \le l < k \le n$, *are concave:*

$$e^{[k,l,\frac{1}{k-l}]}(x+y) \ge e^{[k,l,\frac{1}{k-l}]}(x) + e^{[k,l,\frac{1}{k-l}]}(y).$$

Proof: See [**BMV**], p.306 ▮

Other interesting examples in this class are the scalar curvature ($f = e^{[2,0,1]}$) and the harmonic mean curvature ($f = e^{[n,n-1,1]}$, $\phi = e^{[1,0,-1]}$).

The *power means* provide another class of examples in this category. They are defined as follows for $x = (x_1, \ldots, x_n)$ in Γ_+:

$$\begin{aligned} H_r(x) &= \left(\frac{1}{n}\sum_{i=1}^{n} x_i^r\right)^{\frac{1}{r}} && \text{for } r \ne 0; \\ &= e^{[n,0,\frac{1}{n}]} && \text{for } r = 0. \end{aligned}$$

If $f = H_r$, then $\phi = -H_{-r}^{-1}$. The functions H_r are concave for $r \le 1$, and convex for $r \ge 1$.

More generally, any symmetric function f which is homogeneous of degree one and increasing with respect to each argument gives rise to a family of examples

f_r^α, defined as follows for any r and any nonzero α:

$$f_r^\alpha(x) = \mathrm{sgn}\alpha \cdot \left[\frac{1}{f(1,\ldots,1)} f(x^r)\right]^{\frac{\alpha}{r}} \text{ for } r \neq 0;$$
$$= e^{[n,0,\frac{\alpha}{n}]} \text{ for } r = 0.$$

Homogeneous flows are normally divided into two subsets: The *contraction flows*, for which f is homogeneous of positive degree, and the *expansion flows*, for which f is homogeneous of negative degree.

Non-homogeneous examples : There are many ways to produce functions which are non-homogeneous satisfying the required conditions; most of these are of little interest. However, there are a few examples which have some applications.

The *quasi-arithmetic means* are defined as follows: Suppose $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ is a strictly increasing smooth function on some (possibly infinite) interval I. Define a symmetric function Ψ_n, with $C = I^n$, by $\Psi_n(x) = \Psi^{-1}\left(\frac{1}{n}\sum_{i=1}^n \Psi(x_i)\right)$. Many of the examples already discussed are of this form.

A further useful example is the following: For $k \in \mathbb{R}$, and f a homogeneous example as in the previous section, define $\tilde{f}(x) = f(x - (k,\ldots,k))$, with domain $C' = (k,\infty)^n$. These have some geometrical applications, which are discussed in section VI of this thesis.

4.2 Anisotropic Flows :

Taylor [**CHT**] has considered a class of flows which are anisotropic, as a model for crystal growth phenomena. These flows take the form $S = -\mu(\nu)H$, where H is the mean curvature, and μ is a 'mobility function'; this is usually given as the support function of some fixed convex hypersurface W, called the *Wulff shape.* More generally, one might consider flows of the form $S = -\mu(\nu)f(\lambda)$, for any of the examples of the preceding sections.

A slightly different class of flows is the following, referred to as *relative curvature flows*: Let W be a fixed, strictly convex Wulff shape with support function μ. The Weingarten map $\mathcal{Y}$ of W is then given by lemma (3–14):

$$\mathcal{Y}^{-1} = \bar{g}^*(\mathrm{Hess}_{\bar{\nabla}}\mu + \mu\bar{g})$$

Now consider flows with $S = \mu(\nu)\Phi(\mathcal{Y}^{-1} \circ \mathcal{W})$, where Φ is any example from above. These flows have the desirable property that the Wulff shape evolves trivially—the hypersurfaces at different times are identical up to a scaling factor. Some special examples of such anisotropic flows are used in section IV.

5. Harnack Inequalities

The previous chapters have set up the tools necessary to prove the main results of this section. The Gauss map parametrisation reduces the main result to the following short calculation:

Lemma 5–1. *Suppose φ is a solution to* (1–1) *for which all the hypersurfaces $\varphi_t(M^n)$ are strictly locally convex. In the Gauss map parametrisation, the following evolution equation holds for the quantity $P = \frac{\partial}{\partial t}S$, where we denote $Q = \frac{\partial}{\partial t}A$:*

$$\frac{\partial}{\partial t}P = \bar{\mathcal{L}}P + \dot{\Phi}(\mathrm{Id})P + \ddot{\Phi}(Q,Q) \tag{5–2}$$

where $\ddot{\Phi}(\mathcal{Z}) \in (TS^n \otimes T^*S^n) \otimes (TS^n \otimes T^*S^n)$ is defined by:

$$\ddot{\Phi}(\mathcal{Z})(\mathcal{B},\mathcal{C}) = \frac{\partial}{\partial b}\frac{\partial}{\partial c}\Phi(\mathcal{Z} + b\mathcal{B} + c\mathcal{C})\Big|_{b=c=0} \tag{5–3}$$

for every $\mathcal{B}$ and $\mathcal{C}$ in $T^*S^n \otimes TS^n$.

Proof: Note that $P = \dot{\Phi}(Q)$. Differentiation of equation (3–9) yields the result immediately, since the metric $\bar{g}$ and the connection $\bar{\nabla}$ are independent of time. ∎

Example : For the harmonic mean curvature flow this calculation is as follows:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}S\right) &= \frac{\partial}{\partial t}\left(S^2\left(\bar{\Delta}S + nS\right)\right)\\ &= S^2\left(\bar{\Delta}\left(\frac{\partial}{\partial t}S\right) + n\frac{\partial}{\partial t}S\right) + 2S\left(\frac{\partial}{\partial t}S\right)\left(\bar{\Delta}S + nS\right)\\ &= \bar{\mathcal{L}}\left(\frac{\partial}{\partial t}S\right) + nS^2\left(\frac{\partial}{\partial t}S\right) + \frac{2}{S}\left(\frac{\partial}{\partial t}S\right)^2.\end{aligned}$$

If a function $\Phi : T^*S^n \otimes TS^n \to \mathbb{R}$ satisfies the condition $\ddot{\Phi}(\mathcal{Z})(\mathcal{A},\mathcal{A}) \leq 0$ for all $\mathcal{Z}$ in Ω and all $\mathcal{A}$ in $T^*S^n \otimes TS^n$, then Φ will be called *concave*; if the reverse inequality holds, Φ will be called *convex*. If $\Phi = \mathrm{sgn}\alpha \cdot B^{\alpha}$, where B is positive and concave (convex), then Φ is called *α-concave* (*α-convex*). In terms of the derivatives of Φ, α-concavity is equivalent to the inequality:

$$\ddot{\Phi} \le \frac{\alpha - 1}{\alpha\Phi}\dot{\Phi} \otimes \dot{\Phi}. \tag{5–4}$$

These conditions become considerably more complicated when written in terms of the principal curvatures and the function F. For example, concavity of Φ becomes:

$$\ddot{F}(\mathcal{B}, \mathcal{B}) + 2\dot{F}(\mathcal{B} \circ \mathcal{W}^{-1} \circ \mathcal{B}) \ge 0 \tag{5–5}$$

for every symmetric map $\mathcal{B}$ in $T^*M \otimes TM$.

Theorem 5–6. *Suppose φ is a strictly convex solution to* (1–1).

(1). Suppose Φ is α-concave for $\alpha < 1$. Then the following estimate holds in the Gauss map parametrisation for positive times t, as long as the solution exists:

$$\frac{\partial}{\partial t}S + \frac{\alpha S}{(\alpha - 1)t} \le 0. \tag{5–7}$$

(2). If Φ is positive and concave, then the following weaker estimate holds whenever $t_2 > t_1 > 0$:

$$\sup_{t_2}\left(\frac{\partial}{\partial t}\ln S\right) \le \sup_{t_1}\left(\frac{\partial}{\partial t}\ln S\right). \tag{5–8}$$

(3). Suppose Φ is α-convex, for $\alpha > 1$. Then the following holds at every x in M^n and every $t > 0$:

$$\frac{\partial}{\partial t}S + \frac{\alpha S}{(\alpha - 1)t} \ge 0. \tag{5–9}$$

(4). If Φ is positive and convex, then the following holds whenever $t_2 > t_1$:

$$\inf_{t_2}\left(\frac{\partial}{\partial t}\ln S\right) \ge \inf_{t_1}\left(\frac{\partial}{\partial t}\ln S\right). \tag{5–10}$$

Proof: I will prove the first two cases: The last term in equation (5–2) can be estimated as follows, since Φ is α-concave:

$$\ddot{\Phi}(\mathcal{B},\mathcal{B}) \leq \frac{\alpha - 1}{\alpha\Phi}\left(\dot{\Phi}(\mathcal{B})\right)^2$$

for any $\mathcal{B}$ in $T^*S^n \otimes TS^n$. For the case $\alpha = 1$, the following inequality holds for the quantity $R = \frac{\partial}{\partial t}\ln S$:

$$\frac{\partial}{\partial t}R \leq \bar{\mathcal{L}}R + \frac{2}{S}\dot{\Phi}\left(\bar{g}^*\left(\bar{\nabla}S \otimes \bar{\nabla}R\right)\right).$$

The result (5–8) follows immediately from the parabolic maximum principle, since the first term is an elliptic operator, and the second a gradient term. In the case where $\alpha < 1$, one can estimate as follows, where $R = t\frac{\partial}{\partial t}S + \frac{\alpha S}{\alpha - 1}$:

$$\frac{\partial}{\partial t}R \leq \bar{\mathcal{L}}R + \left(\frac{\alpha - 1}{\alpha S}\frac{\partial}{\partial t}S + \dot{\Phi}(\mathrm{Id})\right)R.$$

Since $t\frac{\partial}{\partial t}S + \frac{\alpha S}{\alpha - 1}$ is initially negative, the parabolic maximum principle implies that it remains so as long as the solution exists. The proof for the convex case is similar. ∎

In the isotropic case, this calculation can immediately be transferred to the standard parametrisation, by writing the various quantities in terms of the metric and connection on the hypersurface. This is most easily done by investigating the change in the evolution equations coming from the modified parametrisation. Here we denote by $I\!I^{-1}$ the map inverse to $I\!I$ in the following sense: $I\!I$ is an element of $T^*M \otimes T^*M$, so we can consider it as a map from TM to T^*M. $I\!I^{-1}$ is then a map from T^*M to TM, and is therefore an element of $TM \otimes TM$.

Corollary 5–11. *Suppose φ is a strictly convex solution to* (1–1).

(1). If Φ is α-concave for some $\alpha < 1$, the following inequality holds in the standard parametrisation:

$$\frac{\partial}{\partial t}S + II^{-1}(\nabla S, \nabla S) + \frac{\alpha S}{(\alpha - 1)t} \leq 0. \tag{5–12}$$

(2). If Φ is concave and positive. then the following holds:

$$\sup_{S^n}\left(\frac{\partial}{\partial t}\ln S + SII^{-1}\left(\nabla \ln S, \nabla \ln S\right)\right) \quad \textit{is decreasing.} \tag{5–13}$$

(3). If Φ is α-convex for $\alpha > 1$. then:

$$\frac{\partial}{\partial t}S + II^{-1}(\nabla S, \nabla S) + \frac{\alpha S}{(\alpha - 1)t} \geq 0. \tag{5–14}$$

(4). If Φ is convex and positive. then:

$$\inf_{S^n}\left(\frac{\partial}{\partial t}\ln S + SII^{-1}\left(\nabla \ln S, \nabla \ln S\right)\right) \quad \textit{is increasing.} \tag{5–15}$$

Proof: The standard parametrisation differs from the Gauss map parametrisation by a diffeomorphism of M which changes in time. This introduces a gradient term into the evolution equations, which can be calculated from (I.3–9), the equation which gives the change in the normal direction under the standard parametrisation. For a function χ we have the following expression which relates the time derivatives in the two settings:

$$\left(\frac{\partial}{\partial t}\chi\right)_{\text{Gauss}} = \left(\frac{\partial}{\partial t}\chi\right)_{\text{standard}} + II^{-1}g^{*}(\nabla S \otimes \nabla \chi). \tag{5–16}$$

In particular, the expression (I.3–12) for the evolution of the speed S in the standard parametrisation (which holds for arbitrary isotropic flows) leads to the following expression for the Gauss map evolution of S :

$$\left(\frac{\partial}{\partial t}S\right)_{\text{Gauss}} = \mathcal{L}S + S\dot{F}(\mathcal{W}^2) + I\!I^{-1}(\nabla S, \nabla S). \tag{5–17}$$

This expression immediately gives the results above from theorem (5–6). ∎

Remark : It is possible to perform the calculations of lemma (5–1) entirely in the standard parametrisation—this was done in the special cases proved in [**Ha3**] and [**Ch3**]. The calculations are then much messier, since the connection and metric are time-dependent, and there are extra gradient terms in the quantities of interest. Furthermore, these calculations were carried out only for isotropic flows; in the case of anisotropic flows the calculations rapidly become unmanageable. This extra complication in the calculations made the simplicity of the results rather mysterious—particularly since the equations are fully nonlinear. The results here are made easier because the evolution equations have a very nice form in the Gauss map parametrisation. This parametrisation of the hypersurfaces seems more geometrically natural for this situation: For example, solutions for which the hypersurfaces evolve by pure scaling (homothetic solutions) have a very simple description in the Gauss map parametrisation, but not in the standard parametrisation. The deep relationship between these homothetic solutions and Harnack inequalities has been noted before ([**Ha3**],[**Ch3**]).

The inequalities of (5–11) can be applied to give a Harnack inequality for the isotropic case:

Theorem 5–18. *Suppose φ is a strictly convex solution of an isotropic equation of form* (1–1). *The following inequalities apply in the standard parametrisation for the cases described, for any points x_1 and x_2 in M^n, any times $t_2 > t_1 > 0$, and any curve γ joining (x_1, t_1) to (x_2, t_2):*

(1). Φ α-concave, $\alpha < 0$:

$$\text{(5–19)} \qquad \frac{S(x_2,t_2)}{S(x_1,t_1)} \geq \left(\frac{t_1}{t_2}\right)^{\frac{\alpha}{\alpha-1}} \exp\left(-\frac{1}{4}\int_\gamma |S|^{-1} I\!I(\dot\gamma,\dot\gamma)\,dt\right).$$

(2). Φ α-convex, $\alpha > 1$:

$$\text{(5–20)} \qquad \frac{S(x_2,t_2)}{S(x_1,t_1)} \geq \left(\frac{t_1}{t_2}\right)^{\frac{\alpha}{\alpha-1}} \exp\left(-\frac{1}{4}\int_\gamma S^{-1} I\!I(\dot\gamma,\dot\gamma)\,dt\right).$$

(3). Φ convex and positive:

$$\text{(5–21)} \qquad \frac{S(x_2,t_2)}{S(x_1,t_1)} \geq \exp\left(-C_1(t_2-t_1) - \frac{1}{4}\int_\gamma S^{-1} I\!I(\dot\gamma,\dot\gamma)\,dt\right)$$

where $C_1 = -\inf_{t=0}\left(\frac{\partial}{\partial t}\ln S + S I\!I^{-1}(\nabla \ln S, \nabla \ln S)\right)$.

Proof: I will give the proof only for one case—the other calculations are similar. Consider the case where Φ is α-concave for $\alpha < 0$: Along a curve γ,

$$D_{\dot\gamma} \ln|S| = \frac{\partial}{\partial t}\ln|S| + \langle \dot\gamma, \nabla \ln|S|\rangle.$$

This can be estimated using (5–12) and the Cauchy-Schwarz inequality:

$$\begin{aligned} D_{\dot\gamma}\ln|S| &\geq |S| I\!I^{-1}(\nabla\ln|S|, \nabla\ln|S|) + \langle\dot\gamma, \nabla\ln|S|\rangle - \frac{\alpha}{(\alpha-1)t} \\ &\geq -\frac{1}{4}S^{-1} I\!I(\dot\gamma,\dot\gamma) - \frac{\alpha}{(\alpha-1)t}. \end{aligned}$$

Integrating along the curve γ yields (5–19). ∎

Example : For the mean curvature flow we have $|S|^{-1} I\!I \leq 1$, and hence the estimate (5–19) becomes

$$\frac{H(x_2,t_2)}{H(x_1,t_1)} \geq \left(\frac{t_1}{t_2}\right)^{\frac{1}{2}} \exp\left(-\frac{d^2}{4(t_2-t_1)}\right)$$

where d is the distance from x_1 to x_2 with respect to the metric g at time t_1.

The estimates in theorem (5–18) have been obtained by Hamilton [**Ha3**] for the mean curvature flow, and by Chow [**Ch3**] for flows by positive powers of the Gauss curvature. It should be noted that the integrals on the right hand side of these inequalities are in general difficult to estimate—for the mean curvature flow, there is a useful estimate as noted above, but for flows which are homogeneous with powers other than one, more natural estimates can be found by integrating the inequalities from (5–11) in a different way. The following theorem summarises these results in the special case where the second fundamental form can be controlled in terms of an appropriate power of the speed. This is automatically the case, for example, for speeds given by powers of the power means H_r with $r > 0$, or for powers of functions for which $\frac{H}{f}$ is bounded above on the positive cone $[0, \infty)^n$.

Examples : In section I it was shown that solutions to a wide class of flows with speeds F homogeneous of degree one satisfy a condition $\frac{H}{F} \leq C$, where C depends on the initial hypersurface. Hence all these solutions satisfy the hypotheses of the following theorem.

Theorem 5–22. *Suppose φ is a strictly convex solution to an isotropic equation of the form (1–1), and the speed satisfies $\Phi = \mathrm{sgn}\alpha \cdot B^\alpha$, for some homogeneous degree 1 function B. Assume further that $B(A)A^{-1} \leq C_2 \mathrm{Id}$ on the solution φ, for some constant C_2. Then the following estimates hold for any times $t_2 > t_1 > 0$ and any points x_1 and x_2 in M^n:*

(1). If B is concave and $\alpha < -1$:

(5–23)
$$t_2^{\frac{\alpha+1}{\alpha-1}} |S(x_2, t_2)|^{\frac{\alpha+1}{\alpha}} - t_1^{\frac{\alpha+1}{\alpha-1}} |S(x_1, t_1)|^{\frac{\alpha+1}{\alpha}} \leq \frac{(\alpha+1) C_2 d^2}{2\alpha(1-\alpha)\left[t_2^{\frac{2}{1-\alpha}} - t_1^{\frac{2}{1-\alpha}}\right]}$$

where d is the distance from x_1 to x_2 with respect to the metric g at time t_1.

(2). If B is concave, and $\alpha = -1$:

$$\text{(5–24)} \qquad \frac{S(x_2,t_2)}{S(x_1,t_1)} \geq \left(\frac{t_1}{t_2}\right)^{\frac{1}{2}} \exp\left(-\frac{C_2 d^2}{4(t_2-t_1)}\right)$$

where d is the same as in (1).

(3). If B is concave and $-1 < \alpha < 0$:

(5–25)

$$t_2^{\frac{\alpha+1}{\alpha-1}} |S(x_2,t_2)|^{\frac{\alpha+1}{\alpha}} - t_1^{\frac{\alpha+1}{\alpha-1}} |S(x_1,t_1)|^{\frac{\alpha+1}{\alpha}} \leq \frac{|1+\alpha| C_2 d^2}{2|\alpha(1-\alpha)| \left[t_2^{\frac{2}{1-\alpha}} - t_1^{\frac{2}{1-\alpha}}\right]}$$

where d is the same as in (1).

(4). If B is convex and $\alpha > 1$:

(5–26)

$$t_2^{\frac{\alpha+1}{\alpha-1}} S(x_2,t_2)^{\frac{\alpha+1}{\alpha}} - t_1^{\frac{\alpha+1}{\alpha-1}} S(x_1,t_1)^{\frac{\alpha+1}{\alpha}} \geq -\frac{(\alpha+1) C_2 d^2}{2\alpha(1-\alpha) \left[t_1^{\frac{2}{1-\alpha}} - t_2^{\frac{2}{1-\alpha}}\right]}$$

where d is the distance between x_1 and x_2 with respect to the metric g at time t_2.

Example : Flows with $F = H^\alpha$ satisfy the required conditions with $C_2 = 1$.

Proof : Consider the case (1). The estimate (5–12) can be written as follows:

$$\frac{\partial}{\partial t}\left(t^{\frac{\alpha+1}{\alpha-1}} |S|^{\frac{\alpha+1}{\alpha}}\right) \geq \frac{\alpha t^{\frac{\alpha+1}{1-\alpha}}}{\alpha+1} (B \cdot I\!I)^{-1} \left(\nabla\left(t^{\frac{\alpha+1}{\alpha-1}} |S|^{\frac{\alpha+1}{\alpha}}\right), \nabla\left(t^{\frac{\alpha+1}{\alpha-1}} |S|^{\frac{\alpha+1}{\alpha}}\right)\right).$$

This can be used in the same manner as in the proof of theorem (5–18) to yield the following inequality:

$$t_2^{\frac{\alpha+1}{\alpha-1}} |S(x_2,t_2)|^{\frac{\alpha+1}{\alpha}} - t_1^{\frac{\alpha+1}{\alpha-1}} |S(x_1,t_1)|^{\frac{\alpha+1}{\alpha}} \geq -\frac{C_2(\alpha+1)}{4\alpha} \int_{t_1}^{t_2} t^{\frac{\alpha+1}{\alpha-1}} g_{t_1}(\dot\gamma, \dot\gamma)\, dt.$$

The result follows by minimising the integral on the right hand side over all paths joining the two points. The calculations for the other cases are similar. ∎

In the case where the flow is anisotropic, Harnack inequalities can still be obtained by using estimates on the extent of anisotropy of the function Φ. The techniques involved are largely the same as those employed here for the isotropic case.

In some instances where Harnack inequalities are obtained, one also obtains an integal estimate or entropy estimate by integrating the evolution equation from lemma (5–1) over the whole manifold M^n. This is true, for example, in the case of the Gauss curvature flow [**Ch3**]. I will now describe an entropy estimate which holds for a class of flows including the Gauss curvature and the harmonic mean curvature. This estimate was pointed out to me by Gerhard Huisken in the case of the harmonic mean curvature flow.

Theorem 5–27. *Suppose Φ is α-concave for some $\alpha > 0$, and the map $\bar{\nabla}\left(\Phi^{-2}\dot{\Phi}\right) : TS^n \otimes T^*S^n \otimes TS^n \to \mathbb{R}$ satisfies the following condition:*

$$\bar{\nabla}\left(\Phi^{-2}\dot{\Phi}\right)(\mathrm{Id}\otimes u) = 0 \tag{5–28}$$

for all $u \in TS^n$. For any strictly convex solution $\{M_t\}$, the following evolution equation holds under an isotropic flow (1–1) *with speed $S = \Phi(A)$:*

$$\frac{\partial}{\partial t}\int_{S^n}\frac{\partial}{\partial t}(\ln|S|)\,d\mu \ge \frac{\alpha+1}{\alpha}\int_{S^n}\left(\frac{\partial}{\partial t}\ln|S|\right)^2 d\mu. \tag{5–29}$$

Consequently the integral can be estimated in terms of the maximum interval of existence T of the solution:

$$\int_{S^n}\left(\frac{\partial}{\partial t}\ln|S|\right)d\mu \le \frac{\alpha}{(\alpha+1)|S^n|T} \tag{5–30}$$

where $|S^n|$ is the volume of the manifold S^n.

The condition (5–28) says that the trace of $\bar{\nabla}(\Phi^{-2}\dot{\Phi})$ over the first two arguments is identically zero—in local coordinates, $\bar{\nabla}_i(\Phi^{-2}\dot{\Phi}^i_j) = 0$.

The entropy estimate (5–30) amounts to a kind of Poincaré inequality for the speed—for example, in the case of the harmonic mean curvature flow:

(5–31) $$n \int_{S^n} (H_{-1})^2 \, d\mu \leq \frac{1}{2|S^n|T} + \int_{S^n} |\bar{\nabla} H_{-1}|^2 \, d\mu.$$

The only reference to the flow in this inequality is in the time of existence T. For a compact convex initial hypersurface this can be computed exactly for these special flows. This calculation is carried out in section IV.

The only isotropic homogeneous speeds which satisfy the condition (5–28) are the following special functions:

Corollary 5–32. *Suppose* $\phi = e^{[k,0,-1]}$ *(and hence also* $f = e^{[n,n-k,1]}$*) for* $k = 1, \ldots, n$. *Then theorem* (5–27) *holds with* $\alpha = k$.

Remark : For $k = 1$ this is the harmonic mean curvature flow; $k = n$ gives the Gauss curvature flow.

Proof of theorem 5–27 : Integrating using equation (5–2) yields the following:

(5–33)
$$\begin{aligned} \frac{\partial}{\partial t} \int_{S^n} \frac{\partial}{\partial t} \ln |S| \, d\mu = & \int_{S^n} S \left(\Phi^{-2} \dot{\Phi} \bar{g}^* \left(\mathrm{Hess}_{\bar{\nabla}} \frac{\partial}{\partial t} S + \bar{g} \frac{\partial}{\partial t} S \right) \right) d\mu \\ & + \int_{S^n} S^{-1} \ddot{\Phi} \left(\bar{g}^* \mathrm{Hess}_{\bar{\nabla}} S + \mathrm{Id} S, \bar{g}^* \mathrm{Hess}_{\bar{\nabla}} S + \mathrm{Id} S \right) d\mu \\ & - \int_{S^n} \left(S^{-1} \frac{\partial}{\partial t} S \right)^2 d\mu. \end{aligned}$$

Now integration by parts yields the following, after using the α-concavity condition and equation (5–28):

(5–34) $$\frac{\partial}{\partial t} \int_{S^n} \left(\frac{\partial}{\partial t} \ln |S| \right) d\mu \geq \frac{\alpha + 1}{\alpha} \int_{S^n} \left(\frac{\partial}{\partial t} \ln |S| \right)^2 d\mu.$$

The result follows by applying Hölder's inequality and comparing with the ordinary differential equation

$$\frac{\partial}{\partial t}x = \frac{\alpha+1}{\alpha}x^2,$$

since the integral must remain finite as long as the solution exists. ∎

In section V of this thesis it is shown that entropy estimates are intimately related to the Aleksandrov-Fenchel inequalities for convex bodies. A wider class of flows with associated entropy estimates (including anisotropic flows) is also given.

6. Complete Hypersurfaces

The previous chapters have been concerned exclusively with compact hypersurfaces. In this chapter we consider the extension of the main results of the section to complete, non-compact hypersurfaces. Note that the initial calculation (5–1) does not depend on compactness at all; the difficulties arise in the application of the parabolic maximum principle to deduce the inequalities of theorem (5–6). Our approach here will be to reduce the problem to compact subsets by multiplying with appropriate functions with compact support. The techniques are based on the work of Ecker and Huisken [**EH**] which produced interior estimates for the mean curvature flow. I will give the calculation for the isotropic case; the general case is rather more complicated but can be handled by the same methods.

The main result of this chapter is the following:

Theorem 6–1. *Let $\varphi : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a solution to equation* (1–1) *with speed $S = \Phi(A) = -F(\mathcal{W})$, where Φ is α-concave for some α less than or equal to -1, and M^n is a complete manifold. Assume that F and $\dot{F}$ are bounded on the region $M \times [0,T)$, and that $|\nabla \ln F| < C\left(1+|\varphi|^2\right)$. Then equation* (5–12) *holds. Consequently the conclusions of theorems* (5–18) *and* (5–22) *also hold.*

Proof : I will prove this by taking a limit of interior estimates over larger and larger regions. These interior estimates are of some interest in their own right.

We must first devise suitable cut-off functions with support in compact regions of Euclidean space. We use the function $\eta = \max\{\left(R^2 - |\varphi|^2\right)^2, 0\}$, where R is a positive constant. The evolution equation for this quantity is given as follows:

$$\frac{\partial}{\partial t}|\varphi|^2 = -2F\langle\varphi,\nu\rangle \tag{6–2}$$
$$= \mathcal{L}|\varphi|^2 - 2\dot{F}(\mathrm{Id}) + 2(\alpha-1)F\langle\varphi,\nu\rangle$$

$$\frac{\partial}{\partial t}\eta = \mathcal{L}\eta - 2\dot{F}g^*(\nabla|\varphi|^2,\nabla|\varphi|^2) \tag{6–3}$$
$$+ 4\eta^{\frac{1}{2}}\left(\dot{F}(\mathrm{Id}) + (1-\alpha)F\langle\varphi,\nu\rangle\right)$$

where α is the degree of homogeneity of F. Thus we have the following inequalities, in view of the boundedness assumptions on F and $\dot{F}$:

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\eta \le C\eta^{\frac{1}{2}} \tag{6–4}$$
$$|\nabla\eta|^2 \le CR^2\eta.$$

We can write equation (5–2) in the standard parametrisation as follows, making use of the equations (5–16), (2–7) and (3–3):

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)P \le \dot{F}(\mathcal{W}^2)P + \frac{(\alpha+1)P^2}{\alpha F} \tag{6–5}$$

where $P = -\mathcal{L}F - F\dot{F}(\mathcal{W}^2) + I\!I^{-1}(\nabla F,\nabla F)$. Now let $Z = \frac{P}{F}$. The following evolution equation applies:

$$(\frac{\partial}{\partial t} - \mathcal{L})Z \le \frac{2}{F}\dot{F}g^*(\nabla F,\nabla Z) - \frac{\alpha+1}{\alpha}Z^2. \tag{6–6}$$

Now multiply by η and consider the resulting evolution equation:

$$(\frac{\partial}{\partial t} - \mathcal{L})(\eta Z) \le 2\dot{F}g^*\left(\nabla\ln\left(\frac{F}{\eta}\right)\otimes(\nabla(\eta Z) - P\nabla\eta)\right) \tag{6–7}$$
$$- \frac{\alpha+1}{\alpha}\eta Z^2 + Z(\frac{\partial}{\partial t} - \mathcal{L})\eta.$$

We can estimate this using the bounds assumed in the statement of the theorem, using the fact $\eta \le R^4$.

(6–8)
$$(\frac{\partial}{\partial t} - \mathcal{L})(\eta Z) \le 2\dot{F}g^*\left(\nabla\ln\left(\frac{F}{\eta}\right)\otimes\nabla(\eta Z)\right) + CZ(R^2+R^3)$$
$$- \frac{\alpha+1}{\alpha}\eta Z^2.$$

This gives the following estimate for the evolution of $t\eta Z$:

$$(6\text{–}9)\qquad \left(\frac{\partial}{\partial t} - \mathcal{L}\right)(t\eta Z) \leq 2\dot{F}g^{*}\nabla \ln\left(\frac{F}{\eta}\right) \otimes (\nabla(t\eta Z)) + CtR^{3} - \frac{\alpha+1}{\alpha}t\eta Z^{2} + R^{4}Z.$$

This quantity is initially zero. At a point and time where a new maximum is attained, we have the left hand side positive, which gives:

$$(6\text{–}10)\qquad \sup_{|\varphi|\leq R} \frac{\alpha+1}{\alpha}t\eta Z \leq CtR^{3} + R^{4}$$

Applying this on a slightly smaller ball $|\varphi| \leq \theta R$ (equivalently, $\eta \geq (1-\theta^{2})^{2}R^{4}$), we obtain:

$$(6\text{–}11)\qquad \sup_{|\varphi|\leq \theta R} Z \leq \frac{\alpha}{\alpha+1}(1-\theta^{2})^{-2}(CR^{-1} + \frac{1}{t}).$$

where C depends on the bounds assumed in the theorem, taken over the ball $|\varphi| \leq R$. Now take the limit $R \to \infty$:

$$(6\text{–}12)\qquad Z \leq \frac{\alpha}{(\alpha+1)t}$$

which is the estimate required. ∎

This proof is framed in rather general terms. For particular flows, it is often possible to significantly weaken the hypotheses. The general technique, however, is unchanged.

Section III

RESULTS FOR

GENERAL FLOWS

1. Contraction to Points

In this chapter we adapt the techniques of Tso [**Ts**] to a very general class of contracting flows. These techniques were originally developed to deal with the Gauss curvature flow. They were also used by Chow [**Ch1**] for flows by arbitrary positive powers of the Gauss curvature. The results of this chapter will be important in section IV: The new proof of the Aleksandrov-Fenchel inequalities given there depends on the convergence of solutions to points for a particular class of equations.

There are several steps involved in the proof of convergence to a point: The short time existence of solutions presents no difficulties for a strictly parabolic flow—the details are not significantly different from the situation in section I. The first real work comes in showing that the solution remains convex. I will present two methods of proving this—one which applies to a very wide class of isotropic flows, and another which allows anisotropic flows, but requires a somewhat more restricted form for the speed. The next step is the application of the techniques of Tso. This gives an estimate on the speed, as long as the solution encloses a ball of positive radius. Only very few conditions are required to prove this estimate. Finally, we show that the solution remains smooth as long as such a ball is contained. The contraction of the solution to a point follows easily from this. The last steps are more difficult in some cases than others; some further structure conditions are necessary.

I will deal with each main step separately, before gathering the results into a theorem at the end (theorem (1–13)). I will restrict my attention to homogeneous flows, since this is the case of greatest interest. The results can still be pushed through for many reasonable flows with inhomogeneous speeds; the techniques are the same in principle as those presented here.

As in the previous sections, we consider a family of hypersurfaces given by immersions $\varphi : M^n \times [0,T) \to \mathbb{R}^{n+1}$. We assume that the initial hypersurface φ_0 is strictly convex, and that φ evolves according to the following evolution equation:

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(x,t) &= -F(\mathcal{W}(x,t),\nu(x,t))\nu(x,t) \\ \varphi(x,0) &= \varphi_0(x). \end{aligned} \tag{1–1}$$

Here we assume F to satisfy the conditions of section II (most importantly monotonicity, which makes equation (1–1) strictly parabolic). It will be convenient to adopt the notation of section II, making use of the support function. The evolution equation of the support function is as follows:

$$\begin{aligned} \frac{\partial}{\partial t}s(z,t) &= \Phi(A[s],z) \\ s(z,0) &= s_0(z), \end{aligned} \tag{1–2}$$

where Φ is related to F as follows: $\Phi(\mathcal{Z}) = -F(\mathcal{Z}^{-1})$ for every positive definite map $\mathcal{Z}$ of TS^n.

Preserving Convexity : The first result I will prove concerns solutions of isotropic flow equations with speeds of a special form. There are some isotropic speed functions of interest which are not in the class considered here; the majority of these are covered by the second method in theorem (1–7).

The following result was pointed out to me by Gerhard Huisken in the case of speeds which are homogeneous of degree one in the principal curvatures.

Theorem 1–3. *Suppose $\Phi = -B^{-\alpha}$, where $\alpha > 0$ and B is homogeneous of degree one and concave. Then φ_t is strictly convex for every t in $[0,T)$. More precisely, the following estimate holds:*

$$\inf_{M\times\{t\}} \lambda_{\min} \geq \inf_{M\times\{0\}} \lambda_{\min} \tag{1–4}$$

for each $t > 0$.

Proof: Note that since A is the inverse of the Weingarten map $\mathcal{W}$, it is sufficient to obtain a bound on the eigenvalues of the map A.

The evolution equation for A is given by (II.3–8):

$$\frac{\partial}{\partial t}A = \bar{g}^*\mathrm{Hess}_{\bar{\nabla}}\Phi + \Phi\mathrm{Id}.$$

This can be expanded using the definitions of the derivative $\dot{\Phi}$ and second derivative $\ddot{\Phi}$ from section II:

$$\frac{\partial}{\partial t}A_{ij} = \dot{\Phi}^{kl}\bar{\nabla}_i\bar{\nabla}_j A_{kl} + \ddot{\Phi}(\bar{\nabla}_i A, \bar{\nabla}_j A) + \bar{g}_{ij}\Phi$$

This can be put into a useful parabolic form by applying a version of Simons's identity:

$$\dot{\Phi}^{kl}\bar{\nabla}_i\bar{\nabla}_j A_{kl} = \dot{\Phi}^{kl}\bar{\nabla}_k\bar{\nabla}_l A_{ij} + \bar{g}_{ij}\dot{\Phi}(A) - A_{ij}\dot{\Phi}(\mathrm{Id}).$$

This gives the following form for the evolution of A:

$$\frac{\partial}{\partial t}A = \bar{\mathcal{L}}A + (1-\alpha)\Phi\mathrm{Id} - \dot{\Phi}(\mathrm{Id})A + \bar{g}^*\ddot{\Phi}(\bar{\nabla}A, \bar{\nabla}A). \tag{1–5}$$

Now we make use of the $(-\alpha)$-concavity of the function Φ. As noted in section II, chapter 5, this implies the following inequality for the derivatives of Φ:

$$\begin{aligned}\ddot{\Phi} &\leq -\frac{\alpha+1}{\alpha|\Phi|}\dot{\Phi}\otimes\dot{\Phi}\\ &\leq 0.\end{aligned}$$

First consider the case $\alpha \in (0,1]$. In equation (1–5), the last term is negative. $1-\alpha$ is positive, so $(1-\alpha)\Phi\mathrm{Id}$ is negative definite. The remaining term is also clearly negative definite. Thus the maximum eigenvalue of A is decreasing, and convexity is preserved.

Next consider the case $\alpha \geq 1$. We can rewrite equation (1–5) as follows:

$$\frac{\partial}{\partial t}A \leq \bar{\mathcal{L}}A - \frac{1}{\alpha}\dot{\Phi}(\mathrm{Id})A - (\alpha-1)\left(\Phi\bar{g} + \alpha^{-1}\dot{\Phi}(\mathrm{Id})A\right). \tag{1–6}$$

The second term here is negative. In order to show the maximum eigenvalue is decreasing, we must show that the last term is negative at the eigenvector with maximum eigenvalue. This is clear, since the last bracket can be estimated as follows at this eigenvector:

$$\begin{aligned}\Phi\mathrm{Id} + \alpha^{-1}\dot{\Phi}A &= \alpha^{-1}\left(\dot{\Phi}(\mathrm{Id})A - \dot{\Phi}(A)\mathrm{Id}\right)\\ &= \alpha^{-1}\sum_{k=1}^{n}\dot{\Phi}^{kk}\left(\kappa_{\max} - \kappa_k\right)\\ &\geq 0\end{aligned}$$

where $\kappa_1, \ldots, \kappa_n$ are the eigenvalues of A (principal radii of curvature). ∎

Examples : All of the classical curvatures fit into this class: We can allow any F of the form $e^{[k,\ell,\alpha]}$ for $0 \leq \ell < k \leq n$ and $\alpha > 0$ (see chapter 4 of section II). Other examples include positive powers of the power means H_r for $r \geq -1$. It is of particular interest to note that there are many flows with speeds F homogeneous of degree one, for which we can show that convexity is preserved. Some of these are not covered by the results of section I; the only step missing for those results to apply is the pinching estimate (see chapter 4, section I).

In section IV, it will be necessary to consider anisotropic flows. The following result shows that convexity is preserved for the flows considered there, and also for a large class of other flows:

Theorem 1–7. *Suppose F is homogeneous of positive degree α, and approaches zero on the boundary of the set S_+ of positive definite maps. Then φ remains strictly convex for all $t > 0$.*

Proof : This theorem is much simpler than the previous one: We need only consider the evolution equation for the speed Φ:

$$\frac{\partial}{\partial t}\Phi = \bar{\mathcal{L}}\Phi + \dot{\Phi}(\mathrm{Id})\Phi.$$

The function Φ is strictly negative, so the second term here is negative. It follows from the parabolic maximum principle that the supremum of Φ is decreasing. Written in terms of F, we have:

$$\inf_t F \geq \inf_0 F. \tag{1–8}$$

Since F approaches zero whenever $\mathcal{W}$ becomes degenerate, this is sufficient to show that strict convexity is preserved. ∎

Examples : This case handles speeds such as the power means H_r for $r < -1$, which were not covered by the previous case. Positive powers of these are also manageable. This theorem also covers a range of anisotropic flows: For example, $F = \mu(\nu)H_r^\alpha$ for $\alpha > 0$ and $r \leq 0$, or $F = \mu(\nu)e^{[n,k,\alpha]}$ for $k < n$ and $\alpha > 0$. Some more complicated anisotropic flows are included as well—see section IV for an important class of examples.

Tso estimates :

Theorem 1–9. *Let φ be a strictly convex solution to* (1–1) *on a time interval* $[0,T)$. *Suppose $\Phi = -B^{-\alpha}$, where B satisfies $\dot{B}(\mathrm{Id}) \geq C_0$ everywhere. If φ encloses a ball of radius $r > 0$ on the entire time interval $[0,T)$, then the speed Φ is bounded above on $[0,T)$.*

Proof : Since the solution φ encloses some ball on a time interval $[0, T)$, an appropriate choice of origin ensures that $s \geq 2\sigma$ on this time interval, for some $\sigma > 0$. Consider the following evolution equation, which can be deduced from (II.3–9) and (1–1):

$$\frac{\partial}{\partial t}\frac{|\Phi|}{s-\sigma} = \bar{\mathcal{L}}\frac{|\Phi|}{s-\sigma} + \frac{2}{s-\sigma}\dot{\Phi}\bar{g}^*\left(\bar{\nabla}s \otimes \bar{\nabla}\left(\frac{|\Phi|}{s-\sigma}\right)\right) + \frac{|\Phi|}{(s-\sigma)^2}\left((1+\alpha)|\Phi| - \sigma\dot{\Phi}(\mathrm{Id})\right). \tag{1–10}$$

The last term can be estimated as follows:

$$\begin{aligned}(1+\alpha)|\Phi| - \sigma\dot{\Phi}(\mathrm{Id}) &= (1+\alpha)B^{-\alpha} - \sigma\alpha B^{-(\alpha+1)}\dot{B}(\mathrm{Id}) \\ &\leq |\Phi|\left(1+\alpha - C_0\sigma\alpha|\Phi|^{\frac{\alpha+1}{\alpha}}\right).\end{aligned}$$

Now substitute this into equation (1–10), noting that $|\Phi| \geq \sigma\frac{|\Phi|}{s-\sigma}$:

(1–11)

$$\left(\frac{\partial}{\partial t} - \bar{\mathcal{L}}\right)\frac{|\Phi|}{s-\sigma} \leq \frac{2}{s-\sigma}\dot{\Phi}\bar{g}^*\left(\bar{\nabla}s \otimes \bar{\nabla}\left(\frac{|\Phi|}{s-\sigma}\right)\right) + \left(\frac{|\Phi|}{s-\sigma}\right)^2\left(1+\alpha - C\alpha\sigma^{\frac{2\alpha+1}{\alpha}}\left(\frac{|\Phi|}{s-\sigma}\right)^{\frac{1+\alpha}{\alpha}}\right).$$

The parabolic maximum principle now gives the following bound:

$$\sup_t\left(\frac{|\Phi|}{s-\sigma}\right) \leq \max\left\{\sup_{t=0}\left(\frac{|\Phi|}{s-\sigma}\right), \left(\frac{1+\alpha}{\alpha C_0}\right)^{\frac{\alpha}{\alpha+1}}\sigma^{-\frac{2\alpha+1}{\alpha+1}}\right\}.$$

Clearly $s - \sigma$ is bounded by the diameter of the initial hypersurface. Hence we have an estimate:

$$|\Phi| \leq C$$

where C depends only on σ, C_0, α, and the diameter of φ_0. ∎

Examples : The condition required for this result is not very restrictive. In particular, if B is concave, we have automatically $\dot{B}(\mathrm{Id}) \geq B(\mathrm{Id}) > 0$. One can get away with even weaker conditions than this—for example, the power means H_r satisfy the condition for any r.

Convergence : Now we must finish the proof of convergence to points. I will show that the bound on the speed $|\Phi|$ is sufficient to give bounds on the entire curvature and the higher derivatives of the curvature. This is an application of the results of Krylov [**K**].

Theorem 1–12. *Suppose φ is a solution to (1–1) on the interval $[0, T)$, which encloses a ball. Assume φ has bounded speed $|\Phi|$, and satisfies a strict convexity condition $A \leq C$. Suppose Φ has the form $\Phi = -B^{-\alpha}$ for some $\alpha > 0$, where B is concave and homogeneous of degree one; assume also that $\dot{B}$ degenerates only where B approaches zero on the boundary of the positive cone. Then the support function s has uniform bounds in C^k for every k on the entire time interval $[0, T)$.*

Proof: The assumption of bounded speed and strict convexity are just sufficient to make the equation uniformly parabolic on this time interval. Note than Φ is concave since B is, and so we can use the results of Krylov ([**K**], section (5.5)) to obtain estimates in $C^{2+\beta}$. Higher estimates then follow from standard Schauder theory. See [**K**] and [**Ts**] for further details of this process. ∎

Remark : It should be noted that this result is by no means exhaustive—for particular flows one can often devise methods which will work. For example, I have not stated this result with sufficient generality to cover anisotropic Gauss curvature flows; however such flows are well-behaved, as shown in [**Ts**]. The same holds for many other anisotropic flows.

The results can be summarised for convenience in the following theorem:

Theorem 1–13. *Let φ_0 be a smooth, strictly convex initial immersion. Suppose Φ has the form $-B^{-\alpha}$, where B is concave and homogeneous of degree one, and either*

(1). Φ is isotropic, or

(2). F approaches zero on the boundary of the positive cone.

Suppose also that $\dot{B}$ degenerates only at points on the boundary of the positive cone where B tends to zero. Then there exists a unique smooth solution of the equation (1–1) *which converges to a point in finite time.*

Examples : The conditions of the theorem allow many useful examples: Isotropic examples include almost any reasonable homogeneous contraction flow with Φ concave, as well as other examples (see the examples after theorem (1–7)). In particular, arbitrary positive powers of classical curvatures make good speeds. Anisotropic examples are more restrictive—in particular, the form of the speed must be such that a bound below on $|\Phi|$ implies a bound below on the smallest eigenvalue of $\mathcal{W}$. Examples include arbitrary anisotropic analogues of the harmonic mean curvature flow, where s evolves by an equation of the following form:

$$\frac{\partial}{\partial t}s = a^{ij}(z)\left(\bar{\nabla}_i\bar{\nabla}_j s + \bar{g}_{ij}s\right).$$

Arbitrary positive powers of such speeds are also acceptable.

2. Contracting Curves

In this chapter I will look closely at evolution equations which contract curves in the plane. Gage and Hamilton [**Ga1-2**], [**GH1**] have shown that a closed convex curve bounding a region of the plane contracts under the curve shortening flow to a point, becoming round as it does so. This is a natural analogue of the result of Huisken for higher dimensional hypersurfaces, but the techniques are quite different. I will extend the work of Gage and Hamilton in two directions: Firstly, I will consider isotropic flows with different homogeneity—flows by a positive power of the curvature. Secondly, I will allow anisotropic flows of a natural type. In all these cases the results of the previous chapter already tell us that solutions converge to points; the difficulty comes in proving that the rescaled curves converge to the expected limit shape. The main tool I will use in all these cases is an generalisation of an isoperimetric inequality due to Gage [**Ga1**].

I have heard recently that Michael Gage has produced a preprint concerning the anisotropic flows of homogeneity one. I have not so far seen this work; however, it seems reasonable to expect that the techniques are similar, since my estimates are a direct analogy of his work in [**Ga1**].

The flows I will consider are the following: We take $s_1 : S^1 \to \mathbb{R}$ to be the support function of a smooth, strictly convex, embedded closed curve in the plane, and evolve a second support function s according to these equations:

$$\frac{\partial}{\partial t}s = -s_1\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)^{\alpha} \tag{2–1}$$
$$s(z,0) = s_0(z)$$

for every z in S^1, where we denote $\mathcal{Q}[f] = f_{zz} + f$ for any function f. Thus $\mathcal{Q}[s]$ is the radius of curvature of the curve given by s. The support function s_1 will be assumed to be symmetric about the origin, so that $s_1(z) = s_1(-z)$ for each z. It is not clear that this assumption is necessary—much weaker conditions suffice to show convergence to a point, for example. We assume the condition because it makes the problem a direct analogue of the isotropic case.

I will prove the following result:

Theorem 2–2. *Let $\alpha \geq 1$. For any smooth initial s_0 given as the support function of a strictly convex embedded curve, there exists a unique smooth solution to the equation (2–1) on a finite time interval $[0, T)$. The solution converges to a point in finite time. After rescaling about the final point to give constant enclosed area, the solution converges smoothly to a multiple of s_1.*

Remark : We consider here only the case $\alpha \geq 1$. There are various other possibilities for flows of curves: The expansion flows will be dealt with in the next chapter, for speeds homogeneous of degree less than or equal to 1 (in the notation of this chapter, $-1 \leq \alpha < 0$). Expansion flows of higher degree ($\alpha < -1$) are treated in chapter 2 of section V, using some new estimates developed there. The contraction flows of small degree ($0 < \alpha < 1$) are also mentioned in section V.

For later convenience we define $V_0 = \int_{S^1} s_1 Q[s_1] d\mu$, $V_1 = \int_{S^1} s Q[s_1] d\mu$, and $V_2 = \int_{S^1} s Q[s] d\mu$. These are natural geometric quantites known as mixed volumes (see section IV). These quantities are related by the Aleksandrov-Fenchel inequality, giving a generalisation of the classical isoperimetric inequality.

The most important part of the proof is the following inequality, which we shall use to show that the isoperimetric ratio improves under the flow.

Lemma 2–3. *For any smooth support function s of a strictly convex bounded region in $\mathbb{R}^2$, and any smooth support function s_1 of a symmetric strictly convex bounded region, the following inequality holds:*

$$\int_{S^1} s_1 \mathcal{Q}[s_1]^2 (\mathcal{Q}[s])^{-1}\, d\mu \geq \frac{V_1 V_0}{V_2}. \tag{2–4}$$

with equality if and only if s is congruent to s_1—that is, there exists a point p in $\mathbb{R}^2$ and a constant $C > 0$ such that $s = Cs_1 + \langle z, p\rangle$.

Remark : This is a precise analogue of the isoperimetric inequality given by Gage in [**Ga1**]. The proof presented below also follows Gage's proof closely.

Proof : As in [**Ga1**], we convert the problem to one about finding a good 'centre' point of a hypersurface. The following proposition states the modified problem and shows its equivalence to the inequality above:

Proposition 2–5. *Suppose for some choice of origin the following inequality holds:*

$$\int_{S^1} s_1 \mathcal{Q}[s_1] \left(\frac{s}{s_1}\right)^2 d\mu \leq \frac{V_1 V_2}{V_0}. \tag{2–6}$$

Then inequality (2–4) *also holds.*

Proof : This follows from an application of the Hölder inequality to the definition of V_1:

$$\begin{aligned} V_1 &= \int_{S^1} s_1 \mathcal{Q}[s_1] \left(\frac{s}{s_1}\right) d\mu \\ &= \int_{S^1} \left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right) \cdot \left(\frac{s}{s_1}\right) .s_1 \mathcal{Q}[s] d\mu \\ &\leq \left(\int_{S^1} s_1 \mathcal{Q}[s] \left(\frac{s}{s_1}\right)^2 d\mu\right)^{\frac{1}{2}} \left(\int_{S^1} s_1 \mathcal{Q}[s] \left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)^2 d\mu\right)^{\frac{1}{2}}. \end{aligned}$$

The second integral here is the same as that in (2–4), while the first is precisely that in (2–6). The proposition follows directly. ▮

We now prove the inequality in the special case of hypersurfaces which are symmetric about the origin, so that $s(z) = s(-z)$ for each z in S^1:

Lemma 2–7. *Let s_1 be the support function of an arbitrary convex hypersurface containing the origin (not necessarily symmetric for the purposes of this lemma). Suppose s is symmetric with respect to s_1, in the sense that $\frac{s(z)}{s_1(z)} = \frac{s(-z)}{s_1(-z)}$. Then the inequality* (2–6) *holds.*

Proof: Define ρ_- and ρ_+ as follows:

$$\rho_- = \sup\left\{\rho : \rho s_1 + \langle z, p\rangle \le s \text{ for some } p \text{ in } \mathbb{R}^2\right\} \tag{2–8}$$

$$\rho_+ = \inf\left\{\rho : \rho s_1 + \langle z, p\rangle \ge s \text{ for some } p \text{ in } \mathbb{R}^2\right\}.$$

These are analogues of the inradius and circumradius (see chapter 2 of section I).

The proof uses a generalised Bonnesen inequality, which states that for any number x between ρ_- and ρ_+, the following holds:

$$V_0 x^2 - 2V_1 x + V_2 \le 0 \tag{2–9}$$

This follows from the Diskant inequalities (see [**BZ**], page 148), which are themselves consequences of the Aleksandrov-Fenchel inequalities. Note that for s satisfying the symmetry condition above, we have $\rho_- \le \frac{s(z)}{s_1(z)} \le \rho_+$ for every z. Hence the inequality (2–9) holds with $x = \frac{s(z)}{s_1(z)}$. Multiplying this inequality by $s_1 \mathcal{Q}[s]$ and integrating over S_1 gives:

$$V_0 \int_{S_1} s_1 \mathcal{Q}[s] \left(\frac{s}{s_1}\right)^2 d\mu \le V_1 V_2$$

which gives the desired result. ∎

Lemma 2–10. *If s_1 is symmetric about the origin, then for any s there is some choice of origin such that the inequality* (2–6) *holds.*

This step is identical to that in [**Ga1**], so I will omit the details. The idea is to consider semicircles in S^1, and show that there is some semicircle which describes a part of the curve containing exactly half the enclosed volume. Reflecting s to opposite semicircles about this axis gives two centrally symmetric regions which have the same area as s; the average of the values of V_1 on these two surfaces gives the value of V_1 for s; and the same applies for the integral in (2–6). Since the inequality holds on each of these regions, it also holds on the non-symmetric region given by s. Note that it is in this step that the symmetry of s_1 is required.

In view of proposition (2–5), this completes the proof of lemma (2–3). ∎

Now we can proceed to apply this to the evolution equations. The beautiful consequence of the inequality (2–4) is the following:

Lemma 2–11. *Under equation* (2–1), *the isoperimetric ratio $\frac{V_1^2}{V_2}$ decreases— strictly unless s is congruent to s_1.*

Remark : A particular case of the Aleksandrov-Fenchel inequalities is the isoperimetric inequality $V_1^2 - V_2 V_0 \geq 0$. This holds with strict inequality unless s is congruent to s_1 (see section IV).

Proof : We calculate the evolution equation of the isoperimetric ratio:

$$\frac{\partial}{\partial t}V_1 = -\int_{S^1} s_1 \mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)^{\alpha} d\mu$$

$$\frac{\partial}{\partial t}V_2 = -2\int_{S^1} s_1 \mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)^{\alpha-1} d\mu$$

$$\frac{\partial}{\partial t}\frac{V_1^2}{V_2} = -\frac{2V_1}{V_2}\left(\int_{S^1} s_1 \mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)^{\alpha} d\mu - \frac{V_1}{V_2}\int_{S^1} s_1 \mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)^{\alpha-1} d\mu\right)$$

First consider the case $\alpha = 1$. Then the second integral in the last expression becomes just V_0, and we recognise the bracket as the quantity in (2–4). Since this is positive, the isoperimetric ratio is decreasing in this case.

Now consider $\alpha > 1$. In this case we have the following trivial consequence of the Hölder inequality:

$$\int_{S^1} s_1 \mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)^{\alpha} d\mu \geq \frac{\int_{S^1} s_1 \mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)^{\alpha-1} d\mu \cdot \int_{S^1} s_1 \mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right) d\mu}{\int_{S^1} s_1 \mathcal{Q}[s_1]\, d\mu}.$$

Rearranging this gives the inequality

$$\frac{\int_{S^1} s_1 \mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)^{\alpha} d\mu}{\int_{S^1} s_1 \mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)^{\alpha-1} d\mu} \geq \frac{\int_{S^1} s_1 \mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right) d\mu}{\int_{S^1} s_1 \mathcal{Q}[s_1]\, d\mu} \geq \frac{V_1}{V_2}.$$

This says precisely that the bracketted term in the evolution equation for $\frac{V_1^2}{V_2}$ is positive. The result follows. ∎

With this information in hand, we can proceed to prove theorem (2–2). Note that the anisotropic Bonnesen inequality enables us to bound the ratio of ρ_+ to ρ_-: The inequality can be written in the form

$$\frac{V_1 - \sqrt{V_1^2 - V_0 V_2}}{V_0} \leq \rho_- \leq \rho_+ \leq \frac{V_1 + \sqrt{V_1^2 - V_0 V_2}}{V_0}.$$

This gives the following estimate for the ratio:

$$\frac{\rho_+}{\rho_-} \leq \frac{\left(V_1 + \sqrt{V_1^2 - V_0 V_2}\right)^2}{V_0 V_2} \leq \frac{4V_1^2}{V_0 V_2} \leq C$$

using the bound on the isoperimetric ratio. The analysis from here on is similar to that in chapter 7 of section I—the Tso estimate gives a uniform bound above on the speed. Since we have a bound on the diameter, we can use the Harnack inequality (Theorem (II.5–18)) to obtain a bound from below on the speed. This makes the equation uniformly parabolic, so we can use the results of Krylov in [**K**] to obtain Hölder estimates on the curvature. Higher regularity follows from Schauder theory.

It follows that we have convergence on a subsequence of times to a smooth curve. The strictly decreasing isoperimetric ratio shows that this limit must be congruent to s_1. Strong convergence follows as in section I. ∎

Remark : This result is the first complete description of the behaviour of flows with higher homogeneity, and also the first complete description of the behaviour of an anisotropic flow. There is good reason to believe that a similar result should hold in higher dimensions—contraction flows with a high degree of homogeneity show some encouraging signs, while those with small degree do not. Some further indications of this can be found in section V, chapter 2.

3. Expansion Flows

In this chapter I will consider expansion flows with speeds homogeneous in the map A, of degree α in the range $(0,1]$. In the isotropic case, these flows have been investigated thoroughly by Huisken [**Hu4**] and Urbas [**U1**] in the case of convex hypersurfaces, and by Gerhardt [**Ge**] and Urbas [**U2**] for star-shaped hypersurfaces. My aim here will be to extend this work to cover flows with anisotropic speeds.

It will be most convenient to perform the analysis on the hypersurface itself, since the Gauss map machinery breaks down. Thus we consider an initial hypersurface $\varphi_0 : M^n \to \mathbb{R}^{n+1}$ which is star-shaped about the origin—that is, the function $\sigma = \langle \Phi, \nu \rangle : M^n \to \mathbb{R}$ is strictly positive everywhere. We consider the evolution of such hypersurfaces under equations of the following form:

$$\begin{aligned} &\frac{\partial}{\partial t}\varphi(x,t) = -S(x,t)\nu(x,t) \\ &\varphi(x,0) = \varphi_0(x) \end{aligned} \tag{3–1}$$

where ν is the outward unit normal at each point, and $\bar{\mathcal{W}}(x,t)$ is a map of $T_{\nu(x,t)}S^n$ given by $\bar{\mathcal{W}}(u) = T\varphi \circ \mathcal{W} \circ T^{-1}\varphi(u)$ for all u in $T_\nu S^n$ (identifying the tangent spaces $T_x\varphi(TM)$ and $T_\nu S^n$). Here the speed S will be given by the following form:

$$S(x,t) = -s_1(\nu(x,t))\left(F(\bar{\mathcal{W}}(x,t))\right)^{-\alpha}.$$

F is a real function defined on a domain of $T^*S^n \otimes TS^n$. We assume that F is smooth, and that, for each z in S^n, F is homogeneous of degree one in $\bar{\mathcal{W}}$, monotonic (so that $\dot{F}$ is positive definite), concave in $\bar{\mathcal{W}}$, and satisfies $F(\bar{\mathcal{W}}_1) = 1$. Here $s_1 : S^n \to \mathbb{R}$ is the support function of a strictly convex hypersurface φ' enclosing the origin, and $\bar{\mathcal{W}}_1$ is the corresponding Weingarten map (given by the inverse of the map $A[s_1]$ in the notation of section II). The domain of definition of F for each z is an open cone in $T^*S^n \otimes TS^n$, containing the cone of positive definite maps. We require that F tends to zero on the boundary of this cone for each z. For simplicity of presentation I will consider only two special cases: In the first case we require the eigenvalues of F to be uniformly bounded above and below. For the second case we require that the domain of definition of F be precisely the set of positive definite matrices, and that the dual function Φ, defined by $\Phi(\mathcal{Z}) = \left(F(\mathcal{Z}^{-1})\right)^{-1}$, is concave and has eigenvalues bounded from above and below.

In the first case the following structure conditions are also required:

$$\begin{aligned} &\dot{F}(\bar{\mathcal{W}}^2) \geq CF^2; \\ &\left|\frac{\partial}{\partial \nu^i} F\right| \leq CF. \end{aligned} \tag{3–2}$$

These conditions on F may be weakened considerably, but the derivation of a curvature estimate becomes messy and complicated. Many flows of interest are included in this class (see the examples below).

I will prove the following result:

Theorem 3–3. *Let $\alpha \in (0,1]$. For any smooth initial hypersurface φ which is star-shaped about the origin and has $\bar{\mathcal{W}}$ in the domain of F, there exists a unique, smooth solution to equation (3–1) on the time interval $[0,\infty)$, which expands to infinity as the final time is approached. If the hypersurfaces are rescaled about the origin to make the enclosed volume constant, the hypersurfaces converge in C^∞ to the limit surface φ'.*

Remark : The expansion flows of higher degree ($\alpha > 1$) display rather different behaviour: Solutions expand to infinite radius in finite time. Although it is difficult to treat this case in as great generality as that presented here, There is good reason to believe that rescaled solutions will still converge to the expected limit shape. I present a proof, for the case of expanding curves in the plane, in chapter 2 of section V.

Examples : This result covers a great variety of evolution equations. Some examples of particular interest have the following forms:

(1). $F(\bar{\mathcal{W}}) = f(\lambda(A[s_1] \circ \bar{\mathcal{W}}))$ for some concave, homogeneous degree one function (see the examples in section I).

(2). $F(\bar{\mathcal{W}}) = \frac{f(\lambda(\bar{\mathcal{W}}))}{f(\lambda(\mathcal{W}_1))}$, with f as in (1).

See also sections IV and V, which deal with an important class of flows slightly different from the examples here.

Proof : Solutions are unique, and exist at least for a short time. This follows by the same argument as in lemma (I.3–6), using the description as a graph over the unit sphere.

Note that we have a good maximum principle which ensures that an enclosed solution remain enclosed: If the distance between the solutions attains a new minimum, the points where this is attained have the same normal direction, but the curvatures at the outer solution are less. The monotonicity of the speed therefore implies that the solutions are moving apart at those points. A result of Hamilton (Ha6) shows that the infinum of the distance is therefore increasing.

We can use this maximum principle to control the radius function $|\varphi|$ as follows: Since φ_0 encloses the origin, there is some r_- such that $r_-\varphi'$ is enclosed by φ_0. The solution with initial condition $r_-\varphi'$ evolves homothetically, and is given by $r_-(t)\varphi'$, where $r_-(t) = \left(r_-^{1-\alpha} + (1-\alpha)t\right)^{\frac{1}{1-\alpha}}$ in the case $\alpha < 1$, and $r_-(t) = r_- e^t$ in the case $\alpha = 1$. This gives a bound from below on $|\varphi|$ for all time. Similarly we have an enclosing copy of φ', given by $r_+(t)\varphi'$. $r_+(t)$ is given by expressions similar to those for $r_-(t)$.

Thus we have estimates $r_-(t) < |\varphi| < r_+(t)$ as long as the solution exists. These estimates are extremely useful, for the following reason: The ratio of r_+ to r_- is bounded. In the case $\alpha = 1$, we have $\frac{r_+(t)}{r_-(t)} = \frac{r_+(0)}{r_-(0)}$. For $\alpha < 1$ we have $\frac{r_+(t)}{r_-(t)} < 1 + O\left(t^{-\frac{1}{1-\alpha}}\right)$, so that in this case we already have uniform convergence of the rescaled hypersurfaces to the expected limit, provided the solution exists for all time.

Now I proceed to the problem of proving existence. The first problem is to prove that the solutions remain star-shaped. Note that if the solution remains convex (that is, the second of the two cases allowed in the definition of F), then σ is comparable to $|\varphi|$, and we already have strong control.

More generally, the required estimate follows by considering the evolution equation of $\sigma = \langle \varphi, \nu \rangle$:

$$\begin{aligned}\frac{\partial}{\partial t}\sigma &= \langle s_1 F^{-\alpha}\nu, \nu\rangle - \langle \varphi, \nabla(s_1 F^{-\alpha})\rangle \\ &= S + \langle \varphi, \partial_\ell\rangle g^{k\ell}\dot{S}^{ij}\nabla_k I\!I_{ij} - \frac{\partial S}{\partial \nu^i}\mathcal{W}^{ij}\langle \varphi, \partial_j\rangle\end{aligned}$$

This equation can be turned into a more useful form by noting the result of applying the elliptic operator $\mathcal{L} = \dot{S}g^*\mathrm{Hess}_\nabla$ to σ:

$$\mathcal{L}\sigma = \alpha S + \langle \varphi, \partial_\ell\rangle g^{k\ell}\dot{S}^{ij}\nabla_k I\!I_{ij} - \dot{S}(\mathcal{W}^2)\sigma.$$

This gives the following evolution equation:

$$\text{(3–4)} \qquad \frac{\partial}{\partial t}\sigma = \mathcal{L}\sigma + \dot{S}(\mathcal{W}^2)\sigma + (1-\alpha)S + \frac{\partial S}{\partial \nu^i}g^{ij}\nabla_j\sigma.$$

This is already enough to show that σ remains strictly positive, so that the solution remains star-shaped. Later we shall prove a more powerful, scaling-invariant estimate.

Some control on the speed S can easily be obtained from the following evolution equation:

$$\frac{\partial}{\partial t}S = \mathcal{L}S + \dot{S}(\mathcal{W}^2) + \frac{\partial S}{\partial \nu^i} g^{ij}\nabla_j S. \tag{3–5}$$

This shows that the supremum of S is decreasing (since S is negative). A better estimate can be obtained by considering the quantity $|S|\,\sigma^{-1}V_{n+1}^{\frac{1-\alpha}{n+1}}$, where V_{n+1} is the enclosed volume of the hypersurface. This quantity is scaling-invariant.

$$\frac{\partial}{\partial t}V_{n+1} = \int_M |S|\, d\mu$$

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\frac{|S|\,V_{n+1}^{\frac{1-\alpha}{n+1}}}{\sigma} &= \frac{2}{\sigma}\dot{S}g^*\left(\nabla\sigma, \nabla\left(\frac{|S|\,V_{n+1}^{\frac{1-\alpha}{n+1}}}{\sigma}\right)\right) \\
&\quad - (1-\alpha)\frac{|S|^2}{\sigma^2}V_{n+1}^{\frac{1-\alpha}{n+1}} + \frac{(1-\alpha)\,|S|}{(n+1)\sigma}V_{n+1}^{-\frac{n+\alpha}{n+1}}\int_M |S|\, d\mu \\
&= \frac{2}{\sigma}\dot{S}g^*\left(\nabla\sigma, \nabla\left(\frac{|S|\,V_{n+1}^{\frac{1-\alpha}{n+1}}}{\sigma}\right)\right) \\
&\quad - (1-\alpha)\frac{|S|}{\sigma}V_{n+1}^{\frac{1-\alpha}{n+1}}\left(\frac{|S|}{\sigma} - \frac{\int_M |S|\, d\mu}{\int_M \sigma\, d\mu}\right).
\end{aligned}$$

Suppose $\frac{|S|}{\sigma}$ attains its maximum at some point x_0. Then we have the inequality $|S(x)|\,\sigma(x_0) \le |S(x_0)|\,\sigma(x)$ for every point x of M. Integrating this over M gives the following inequality:

$$\frac{|S(x_0)|}{\sigma(x_0)} \ge \frac{\int_M |S|\, d\mu}{\int_M \sigma\, d\mu}.$$

It follows that the entire right hand side of the evolution equation above is non-positive at x_0. A maximum principle of Hamilton [**Ha1**] shows that the supremum of $\frac{|S|}{\sigma}V_{n+1}^{\frac{1-\alpha}{n+1}}$ is decreasing. A similar argument shows that the infinum is increasing.

Hence we have proved that there exist constant C_1 and C_2 such that

(3–6)
$$0 < C_1 \leq \frac{|S|}{\sigma} V_{n+1}^{\frac{1-\alpha}{n+1}} \leq C_2 < \infty.$$

Now we return to the problem of obtaining a scaling-invariant estimate on the function σ. Note that σ is always bounded above by $|\varphi|$; it remains to bound σ from below. We consider the quantity $\frac{|\varphi|^2}{\sigma^2}$:

(3–7)
$$\begin{aligned}\frac{\partial}{\partial t}|\varphi|^2 &= \mathcal{L}|\varphi|^2 - 2\dot{S}(\mathrm{Id}) + 2(1+\alpha)\sigma|S| \\ \left(\frac{\partial}{\partial t} - \mathcal{L}\right)\frac{|\varphi|^2}{\sigma^2} &= \frac{2}{\sigma^2}\dot{S}^{ij}\nabla_i\sigma^2\nabla_j\left(\frac{|\varphi|^2}{\sigma^2}\right) \\ &\quad - \frac{2}{\sigma^2}\dot{S}(\mathrm{Id}) + 2(\alpha+1)\sigma^{-1}|S| - \frac{2|\varphi|^2}{\sigma^{-2}}\dot{S}(\mathcal{W}^2) \\ &\quad + 2\frac{|\varphi|^2}{\sigma^4}\dot{S}^{ij}\nabla_i\sigma\nabla_j\sigma - (1-\alpha)\frac{2|\varphi|^2}{\sigma^3}|S| - \frac{2|\varphi|^2}{\sigma^4}\frac{\partial S}{\partial\nu^i}g^{ij}\nabla_j\sigma^2\end{aligned}$$

At a maximum of this quantity, this expression may be significantly simplified by considering the vanishing of the gradient:

(3–8)
$$\nabla_i\left(\frac{|\varphi|^2}{\sigma^2}\right) = \frac{2|\varphi|^2}{\sigma^3}\left(\frac{\sigma}{|\varphi|^2}\delta_i^j - \mathcal{W}_i^j\right)\langle\varphi, \partial_j\rangle.$$

It follows that $\langle\varphi, \partial_j$ must vanish for all eigenvectors ∂_j of $\mathcal{W}$, except possibly those at which the eigenvalue is exactly $\frac{\sigma}{|\varphi|^2}$. Hence we have the following inequality at a point where a maximum is attained, calculating in local coordinates which diagonalise $\mathcal{W}$ at this point:

(3–9)
$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\varphi|^2}{\sigma^2}\right) &\leq -\frac{2}{\sigma^2}\dot{S}(\mathrm{Id}) + \frac{2}{\sigma^2|\varphi|^2}\dot{S}^{ij}\langle\varphi,\partial_i\rangle\langle\varphi,\partial_j\rangle \\ &\quad - \frac{2|\varphi|^2}{\sigma^2}\dot{S}(\mathcal{W}^2) + 2(\alpha+1)\frac{|S|}{\sigma} - \frac{4}{\sigma^2}\frac{\partial S}{\partial\nu^i}g^{ij}\langle\varphi,\partial_j\rangle.\end{aligned}$$

The sum of the first two terms is negative. The term involving $\frac{|S|}{\sigma}$ is bounded by the estimate (3–6):

$$\begin{aligned}\frac{|S|}{\sigma} &\leq C_2 V_{n+1}^{-\frac{1-\alpha}{n+1}} \\ &\leq C\,|\varphi|^{-(1-\alpha)}.\end{aligned}$$

The last term, involving the derivatives of S on the sphere, is bounded as follows, using the assumed structure conditions (3–2):

$$\begin{aligned}\frac{4}{\sigma^2}\frac{\partial S}{\partial \nu^i} g^{ij}\langle \varphi, \partial_j \rangle &\leq \frac{C}{\sigma^2}\,|S|^{\frac{\alpha+1}{\alpha}}\left|\dot{F}(\mathcal{W}^2)\right|^{\frac{1}{2}}|\varphi| \\ &\leq C\left(\frac{|\varphi|}{\sigma}\right)^{1-\frac{1-\alpha}{\alpha}}\dot{S}(\mathcal{W}^2)\end{aligned}$$

where we have used the inequality $\dot{F}(\mathcal{W}^2) \geq C\,|S|^{-\frac{1}{\alpha}} \geq C\sigma^{-\frac{1}{\alpha}}\,|\varphi|^{\frac{1-\alpha}{\alpha}}$, which follows from the definition of S and the estimate (3–6), since $V_{n+1}^{\frac{1}{n+1}}$ is comparable to $|\varphi|$. It is clear from these estimates that if the maximum of $\frac{|\varphi|}{\sigma}$ is large enough, then the last two terms in equation (3–9) can be controlled using the third-last term. This gives a scaling-invariant estimate of the form $\frac{|\varphi|}{\sigma} \leq C$.

Note that we now have very good control on the quantities σ, $|\varphi|$, and S— in particular, the ratios of σ, $|\varphi|$ and $|S|^{\frac{1}{\alpha}}$ are all uniformly bounded. Now we require a bound on the curvature. If the eigenvalues of F are uniformly bounded above and below, we now have a uniformly parabolic equation, and the required estimates follow from standard results. The same holds if the eigenvalues of the dual function Φ are bounded above and below—simply consider the Gauss map parametrisation of section II.

This completes the proof—the uniform estimates ensure that, on a subsequence of times, the solutions converge to a smooth limit; however, we have a strictly decreasing quantity: $\frac{r_+}{r_-}$ strictly decreases unless φ is coincident with the enclosing and enclosed copies of φ'. Thus $r_+ = r_- = 1$, and $\varphi = \varphi'$. This decreasing quantity also shows that the convergence follows for all times, and not just for a subsequence. ∎

Section IV

ALEKSANDROV-FENCHEL

INEQUALITIES

1. Introduction

In 1936 A.D. Aleksandrov ([**Al1**],[**Al2**]) and W. Fenchel [**Fe**] independently proved a fundamental inequality relating the mixed volumes of convex regions of Euclidean space. The consequences of this inequality include the classical isoperimetric inequality and other inequalities involving the integral cross-sectional measures of a convex region.

This section provides a simple new proof of the Aleksandrov-Fenchel inequalities, by deforming convex regions using special parabolic evolution equations. These equations are examples from the broad class of evolution equations considered in section II, and are defined in terms of elliptic operators naturally associated with the mixed volumes. The same flows are the subject of section V.

In chapter 2 the mixed volumes and the Aleksandrov-Fenchel inequalities are introduced. The mixed volumes are initially defined using an elegant geometric recipe. I use this to obtain expressions as integrals over the sphere in terms of support functions, which at first sight seem rather complicated. This situation is greatly improved after some further calculations which expose the beautiful and very useful structure of the integrands. Some special cases of the mixed volumes are the more familiar mean cross-sectional volumes mentioned in the introduction of the thesis.

The Aleksandrov-Fenchel inequalities are simple to state in their most basic form. I also state a few of their interesting consequences, which include the isoperimetric inequality and a wide variety of other useful inequalities.

In chapter 3 the new proof of the inequalities is given. This is quite short—remarkably so in comparison with other proofs of the inequalities. The evolution equations are defined using the structure of the mixed volumes from chapter 2. They are anisotropic flows, homogeneous of degree one in the curvature. It is easily seen that the methods of chapter 1,section III can be applied; hence solutions converge to points in finite time. The Aleksandrov-Fenchel inequalities then follow directly from the evolution equations for the mixed volumes.

In chapter 4 it is shown that slightly different evolution equations can be used to give direct proofs of the most important consequences of the Aleksandrov-Fenchel inequalities, such as the isoperimetric inequality and other inequalities between the mean cross-sectional volumes.

2. Mixed Volumes and the Aleksandrov-Fenchel inequalities

In this chapter we introduce the mixed volumes and explain some of their structure and properties. We also state the Aleksandrov-Fenchel inequalities and some of their most useful consequences. For a more detailed exposition of this material, see [**BZ**].

Mixed Volumes : We will be considering bounded convex regions in Euclidean space. To describe these we use the support function of the boundary, with the machinery introduced in section II. The mixed volumes are most easily defined by looking at Minkowski sums of convex regions: For any two convex regions D_1 and D_2, we can take the Minkowski sum D_1+D_2 given by $\{a+b : a \in D_1, b \in D_2\}$. This sum is again a convex region. The support functions behave very simply under such an operation: If D_1 and D_2 have support functions s_1 and s_2 respectively, then $D_1 + D_2$ has support function s given by

$$s(z) = s_1(z) + s_2(z). \tag{2–1}$$

The volume $V(D)$ of a convex region D can be calculated in terms of the support function, using the following integral over S^n:

$$V(D) = \frac{1}{n+1}\int_{S^n} s \det\left(A[s]\right) d\mu \tag{2–2}$$

where $d\mu$ is the standard measure on S^n. The integrand here is a homogeneous polynomial of degree $n+1$ in s and its second derivatives.

Now we consider a linear combination of convex regions (in the Minkowski sense): Let D_i, $i = 1, \dots, N$ be convex regions with support functions s_i, and consider the Minkowski sum $\sum_{i=1}^{N} \epsilon_i D_i$ for arbitrary positive ϵ_i. The expressions (2–1) and (2–2) show that the volume $V(\sum \epsilon_i D_i)$ is a homogeneous polynomial of degree $n+1$ in the variables ϵ_i:

$$V(\sum \epsilon_i D_i) = \frac{1}{n+1} \sum_{1 \le i_0, \dots, i_n \le N} \epsilon_{i_0} \dots \epsilon_{i_n} V(D_{i_0}, \dots, D_{i_n}) \tag{2–3}$$

where the coefficient $V(D_{i_0}, \dots, D_{i_n})$ is called the *mixed volume* of the $n+1$ regions $D_{i_0}, \dots, D_{i_n}$, and is given by the following expression in local coordinates:

(2–4)

$$V(D_0, \dots, D_n) = \int_{S^n} s_0 \mathcal{Q}[s_1, \dots, s_n]\, d\mu$$

$$\mathcal{Q}[f_1, \dots, f_n] = \frac{1}{n!} \sum_{\sigma, \tau \in S_n} \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma) A[f_1]^{\sigma(1)}_{\tau(1)} \dots A[f_n]^{\sigma(n)}_{\tau(n)}. \tag{2–5}$$

The apparent asymmetry of the expression (2–4) will be explained in lemma (2–12). The operator $\mathcal{Q}$ is a multilinear operator acting on n functions on S^n. It has several important properties:

Lemma 2–6.

(1). $\mathcal{Q}$ is independent of the order of its arguments:

$$\mathcal{Q}[f_1, \dots, f_n] = \mathcal{Q}[f_{\sigma_1}, \dots, f_{\sigma_n}]$$

for every permutation $\sigma \in S_n$.

(2). $\mathcal{Q}[f_1, \dots, f_n]$ is positive for any n functions $f_1, \dots, f_n$ with each $A[f_i]$ positive definite.

(3). If $f_2, \dots, f_n$ are fixed smooth functions such that $A[f_i]$ is positive definite for each i, then $\mathcal{Q}[f] := \mathcal{Q}[f, f_2, \dots, f_n]$ is a nondegenerate second-order linear elliptic operator, given in local coordinates by an expression

of the following form:

$$Q[f] = \sum_{i,j} \dot{Q}^{ij} \left(\bar{\nabla}_i \bar{\nabla}_j f + \bar{g}_{ij} f \right) \tag{2–7}$$

where $\dot{Q}$ is a positive definite matrix at each point of S^n, depending only on the functions $f_2, \ldots, f_n$.

(4). The following identity holds for any $f_2, \ldots, f_n$ as above:

$$\sum_i \bar{\nabla}_i \dot{Q}^{ij} = 0. \tag{2–8}$$

Proof: The proof of (1) is clear from the expression (2–5) above. The claims (2) and (3) can be proved together by induction: Denote by $Q^{(k)}$ an operator which acts on k positive maps $A^{(1)}, \ldots, A^{(k)}$ of a k-dimensional Euclidean space according to the formula

$$Q^{(k)}\left(A^{(1)}, \ldots, A^{(k)}\right) = \sum_{\sigma,\tau \in S_k} \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma) \left(A^{(1)}\right)^{\sigma(1)}_{\tau(1)} \cdots \left(A^{(k)}\right)^{\sigma(k)}_{\tau(k)}. \tag{2–9}$$

Then we have $Q[f_1, \ldots, f_n] = Q^{(n)}(A[f_1], \ldots, A[f_n])$, where we consider $Q^{(n)}$ acting on maps of the n-dimensional Euclidean space TS^n. The positivity of $Q^{(1)}$ is clear, since we have $Q^{(1)}\left(A^{(1)}\right) = \left(A^{(1)}\right)^1_1$. Suppose we know that $Q^{(k)}$ is positive acting on any k positive definite maps of $\mathbb{R}^k$. Choosing a basis $\{e_1, \ldots, e_{k+1}\}$ which diagonalises $A^{(k+1)}$, we find:

(2–10)

$$Q^{(k+1)}\left(A^{(1)}, \ldots, A^{(k+1)}\right) = \sum_{\substack{1 \le i \le k+1 \\ \sigma,\tau \in S_k^{(i)}}} \left(A^{k+1}\right)^i_i \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \prod_{j=1}^{k} \left(A^{(j)}\right)^{\sigma(j)}_{\tau(j)}$$

where we use the notation $S_k^{(i)}$ to denote the set of permutations from $\{1, \ldots, k\}$ to $\{1, \ldots, i-1, i+1, \ldots, k+1\}$. This can be rewritten as follows:

$$Q^{(k+1)}\left(A^{(1)}, \ldots, A^{(k+1)}\right) = \sum_{i=1}^{k+1} \left(A^{(k+1)}\right)^i_i Q^k \left(A^{(1)}\Big|_{e_i^\perp}, \ldots, A^{(k)}\Big|_{e_i^\perp} \right). \tag{2–11}$$

Here $A^{(\ell)}\Big|_{e_i^\perp}$ is the positive operator on the complement $e_i^\perp$ of e_i in $\mathbb{R}^{k+1}$ given by restricting the components of the operator $A^{(\ell)}$. The induction is now clear. Furthermore, this gives an expression for the diagonal elements of the derivative map $\dot{\mathcal{Q}}$: The diagonal element in a direction $\xi \in TS^n$ is precisely the value of $\mathcal{Q}^{n-1}$ acting on the complement of ξ in TS^n, which is positive by the same induction as above.

The proof of (4) follows from manipulation of the definition of $\mathcal{Q}$, using the Codazzi equations $\bar{\nabla}_i A[f]_j^k = \bar{\nabla}_j A[f]_i^k$:

$$\begin{aligned}
\bar{\nabla}_i \dot{\mathcal{Q}}_j^i &= \sum_{\sigma,\tau} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\delta_j^{\sigma(1)}\delta^i_{\tau(1)} \sum_{m=2}^{n} \left(\prod_{\ell \neq m} A[f_\ell]_{\tau(\ell)}^{\sigma(\ell)} \right) \cdot \bar{\nabla}_i A[f_m]_{\tau(m)}^{\sigma(m)} \\
&= \sum_{\sigma,\tau} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\delta_j^{\sigma(1)} \sum_{m=2}^{n} \left(\prod_{\ell \neq m} A[f_\ell]_{\tau(\ell)}^{\sigma(\ell)} \right) \cdot \bar{\nabla}_{\tau(1)} A[f_m]_{\tau(m)}^{\sigma(m)} \\
&= \sum_{\substack{m\geq 2\\ \sigma,\tau}} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)\delta_j^{\sigma(1)} \prod_{\ell \neq m} A[f_\ell]_{\tau(\ell)}^{\sigma(\ell)} \frac{\bar{\nabla}_{\tau(1)} A[f_m]_{\tau(m)}^{\sigma(m)} - \bar{\nabla}_{\tau(m)} A[f_m]_{\tau(1)}^{\sigma(m)}}{2} \\
&= 0.
\end{aligned}$$

∎

These properties of $\mathcal{Q}$ allow us to deduce some important properties of the mixed volumes:

Lemma 2–12. *Let $D_0, D_0', D_1, \ldots, D_n$ be bounded convex regions with support functions $s_0, s_0', s_1, \ldots, s_n$. Then we have the following facts:*

(1). Symmetry: Let σ be a permutation of the set $\{0, \ldots, n\}$. Then

$$V(D_0, \ldots, D_n) = V(D_{\sigma(0)}, \ldots, D_{\sigma(n)}).$$

(2). Invariance under translations: If p is any point of $\mathbb{R}^{n+1}$, then

$$V(D_0 + p, D_1, \ldots, D_n) = V(D_0, D_1, \ldots, D_n).$$

(3). Positivity:

$$V(D_0, \ldots, D_n) \geq 0.$$

(4). Monotonicity: Suppose $D_0 \subseteq D_0'$. Then

$$V(D_0, D_1, \ldots, D_n) \leq V(D_0', D_1, \ldots, D_n).$$

Proof : To prove symmetry, first note that the symmetry of $\mathcal{Q}$ (lemma (2–6), item (1)) shows that $V(D_0, D_1, \ldots, D_n)$ is invariant under interchange of the regions $D_1, \ldots, D_n$. Therefore it is sufficient to show $V(D_0, D_1, D_2, \ldots, D_n) = V(D_1, D_0, D_2, \ldots, D_n)$. This follows from item (4) of lemma (2–6):

$$\begin{aligned}
V(D_0, D_1, \ldots, D_n) &= \int_{S^n} s_0 \mathcal{Q}[s_1]\, d\mu \\
&= \int_{S^n} s_0 \dot{\mathcal{Q}}^{ij} \left(\bar\nabla_i \bar\nabla_j s_1 + \bar g_{ij} s_1\right)\, d\mu \\
&= -\int_{S^n} \bar\nabla_i s_0 \dot{\mathcal{Q}}^{ij} \bar\nabla_j s_1\, d\mu + \int_{S^n} \dot{\mathcal{Q}}^{ij} \bar g_{ij} s_0 s_1\, d\mu \\
&= \int_{S^n} \dot{\mathcal{Q}}^{ij} \left(\bar\nabla_j \bar\nabla_i s_0 + \bar g_{ij} s_0\right) s_1\, d\mu \\
&= \int_{S^n} s_1 \mathcal{Q}[s_0]\, d\mu \\
&= V(D_1, D_0, D_2, \ldots, D_n).
\end{aligned}$$

A similar calculation proves the invariance under translations: First note that for any point p, $A[\langle p, z\rangle] = 0$, using the definition of the second fundamental form of the sphere:

$$\begin{aligned} A[\langle p, z\rangle] &= \bar{\nabla}_i\bar{\nabla}_j\langle p, z\rangle + \bar{g}_{ij}\langle p, z\rangle \\ &= \langle p, \bar{\nabla}_i\bar{\nabla}_j z + \bar{g}_{ij} z\rangle \\ &= \langle p, -I\!I(S^n)_{ij} z + \bar{g}_{ij} z\rangle \\ &= 0 \end{aligned}$$

since $I\!I(S^n) = \bar{g}$. Now we can apply this:

$$\begin{aligned} V(D_0 + p, D_1, \ldots, D_n) &= \int_{S^n} (s_0 + \langle p, z\rangle)\mathcal{Q}[s_1]\, d\mu \\ &= V(D_0, D_1, \ldots, D_n) + \int_{S^n} \langle p, z\rangle \mathcal{Q}[s_1]\, d\mu \\ &= V(D_0, D_1, \ldots, D_n) + \int_{S^n} s_1 \mathcal{Q}[\langle p, z\rangle]\, d\mu \\ &= V(D_0, D_1, \ldots, D_n) \end{aligned}$$

since $A[\langle p, z\rangle] = 0$.

I will next prove (3): From lemma (2–6) we know that $\mathcal{Q}[s_1, \ldots, s_n]$ is positive everywhere. By translation invariance we can assume that D_0 contains the origin, so that s_0 is strictly positive. The integral $\int s_0 \mathcal{Q}[s_1, \ldots, s_n] d\mu$ is therefore also positive.

The monotonicity condition is similar: First note that $s_0' - s_0$ is positive since D_0' contains D_0. The difference in the mixed volumes is given as follows:

$$V(D_0', D_1, \ldots, D_n) - V(D_0, D_1, \ldots, D_n) = \int_{S^n} (s_0' - s_0)\mathcal{Q}[s_1, \ldots, s_n] d\mu$$

which is clearly positive since both terms in the integrand are positive. ∎

It is interesting to note certain special cases of mixed volumes: For any strictly convex region D, define V_k to be the mixed volume of k copies of D with $n - k + 1$

copies of the unit ball B, for $k = 1, \ldots, n+1$:

$$V_k(D) = V(\underbrace{D, \ldots, D}_{k \text{ times}}, \underbrace{B, \ldots, B}_{n+1-k \text{ times}}) \tag{2–13}$$

For $k = n+1$ this gives the volume $V(D)$. For $k = n$ we have $V_n(D) = H^n(\partial D)$, the Hausdorff n-measure of the boundary of D. For $k < n$ these mixed volumes are called the mean cross-sectional volumes of D, and give the average measure of projections of D onto k-planes.

Example : The anisotropic energy function mentioned in the introduction of this thesis is another example of a mixed volume:

$$V(\underbrace{D, \ldots, D}_{n \text{ times}}, D_1) = \int_{\partial D} s_1(\nu)\, d\mu \tag{2–14}$$

where $s_1 : S^n \to \mathbf{R}$ is the support function of the region D_1. Such anisotropic energies have been considered by Almgren and Taylor [**AT**], Taylor [**Ta1–2**], White [**Wh**] and others.

The Aleksandrov-Fenchel inequalities : The Aleksandrov-Fenchel inequalities relate the different mixed volumes which can be formed from a collection of convex regions:

Theorem 2–15 (Aleksandrov, Fenchel). *Let $D_0, \ldots, D_n$ be convex regions in $\mathbb{R}^{n+1}$. Then the following inequality holds:*

(2–16)

$$V(D_0, D_0, D_2, \ldots, D_n)\, V(D_1, D_1, D_2, \ldots, D_n) \le V(D_0, D_1, D_2, \ldots, D_n)^2 .$$

This basic inequality has many useful consequences: For the special case of the integral cross-sectional volumes $V_k(D)$, we have the following:

$$V_k(D)^2 \ge V_{k-1}(D) V_{k+1}(D) \tag{2–17}$$

for $k = 1, \ldots, n$, where we interpret $V_0(D) = V(B)$, the volume of the unit ball. By applying this several times, we obtain the more general inequality

$$V_k(D)^{a+b} \geq V_{k-a}(D)^b V_{k+b}(D)^a \tag{2–18}$$

for $0 \leq k - a < k < k + b \leq n + 1$. In particular, we have the following inequalities:

$$V_k(D)^l \geq V(B)^{l-k} V_l(D)^k \tag{2–19}$$

for $0 < k < l \leq n + 1$. The case $k = n, l = n + 1$ is the isoperimetric inequality.

One can write the Aleksandrov-Fenchel inequalities using the expressions developed above for the mixed volumes:

$$\int_{S^n} s_0 \mathcal{Q}[s_0] d\mu . \int_{S^n} s_1 \mathcal{Q}[s_1] d\mu \leq \left(\int_{S^n} s_0 \mathcal{Q}[s_1] d\mu \right)^2 \tag{2–20}$$

where $\mathcal{Q}$ is the operator given by (2–5) using the support functions $s_2, \ldots, s_n$ of the regions $D_2, \ldots, D_n$. For a fixed s_1 given by the support function of a convex region, this can be rewritten as a Poincaré inequality for arbitrary functions f on S^n:

$$\int_{S^n} \dot{\mathcal{Q}}^{ij} \bar{\nabla}_i \tilde{f} . \bar{\nabla}_j \tilde{f} \, d\mu \geq \int_{S^n} \dot{\mathcal{Q}}^{ij} \bar{g}_{ij} (\tilde{f})^2 \, d\mu$$

where $\tilde{f}$ is obtained from f by the following expression:

$$\tilde{f} = f - \frac{\int_{S^n} f \mathcal{Q}[s_1] d\mu}{\int_{S^n} s_1 \mathcal{Q}[s_1] d\mu} s_1.$$

3. Proof of the Aleksandrov-Fenchel Inequalities

I will consider special parabolic evolution equations which are defined using the notation developed above for the Aleksandrov–Fenchel inequalities. These flows are easiest to describe in terms of the support function of the region D:

$$\begin{aligned} \frac{\partial}{\partial t} s(z,t) &= -\frac{s_1 \mathcal{Q}[s_1]}{\mathcal{Q}[s]} \\ s(z,0) &= s_0(z) \end{aligned} \tag{3–1}$$

for each z in S^n and each time t. Here we consider s_1 to be the support function of a fixed strictly convex region D_1, and $\mathcal{Q}$ to be specified by equation (2–5) in terms of the support functions of fixed strictly convex regions $D_2, \ldots, D_n$. Equation (3–1) is a fully nonlinear, strictly parabolic evolution equation, with a form consistent with the conditions of section II. In the special case where $D_i = B$ for $i = 1, \ldots, n$ this is precisely the flow by harmonic mean curvature. The equation in the general case can be considered an anisotropic analogue of the harmonic mean curvature flow.

The following theorem is a special case of theorem (III.1–13):

Theorem 3–2. *For any initial data s_0 given by the support function of a smooth strictly convex region D_0, there exists a unique, smooth, strictly convex solution s to equation* (3–1) *on a finite time interval* $[0, T)$. *As the final time T is approached, the function s converges uniformly to the support function of a single point.*

Proof : The conditions required for theorem (III.1–13) are satisfied by these flows: First, to preserve convexity, we note that $\dot{\mathcal{Q}}$ is strictly positive definite on

S^n, and so is comparable with the identity. Hence the speed is bounded above and below by multiples of the harmonic mean curvature; since the speed is increasing, the minimum eigenvalue of $\mathcal{W}$ is bounded below.

The next step—the application of Tso's estimate—depends on a bound below on $\dot{\mathcal{Q}}(\mathrm{Id})$, which is clear from the definition. This gives bounds above and below on the speed as long as a ball is contained; this ensures that $\dot{\Phi}$ is bounded above and below, and the Krylov estimates apply. ∎

The proof of the inequality is now very simple: We consider the evolution of the mixed volumes $V(D, D_1, D_2, \ldots, D_n)$ and $V(D, D, D_2, \ldots, D_n)$ under equation (3–1). For convenience we denote these by V_1 and V_2 respectively, and the constant $V(D_1, D_1, D_2, \ldots, D_n)$ by V_0.

The evolution of V_2 can be calculated as follows:

$$
\begin{aligned}
\frac{\partial}{\partial t}V_2 =& \frac{\partial}{\partial t}\int_{S^n} s\mathcal{Q}[s]d\mu \\
=& \int_{S^n}\left((\frac{\partial}{\partial t}s)\mathcal{Q}[s] + s\mathcal{Q}[\frac{\partial}{\partial t}s]\right)d\mu \\
=& 2\int_{S^n}(\frac{\partial}{\partial t}s)\mathcal{Q}[s]d\mu \\
=& -2\int_{S^n} s_1\mathcal{Q}[s_1]d\mu \\
=& -2V_0
\end{aligned}
$$

where we have used the identity (2–8) to integrate by parts between the second and third lines. Hence V_2 decreases at a uniform rate. Next we calculate the evolution of V_1:

$$\begin{aligned}\frac{\partial}{\partial t}V_1 =&\frac{\partial}{\partial t}\int_{S^n} sQ[s_1]d\mu\\ =&-\int_{S^n} s_1Q[s_1]\left(\frac{Q[s_1]}{Q[s]}\right)d\mu\\ \leq&-\frac{\left(\int_{S^n} s_1Q[s_1]d\mu\right)^2}{\int_{S^n} s_1Q[s_1]\left(\frac{Q[s]}{Q[s_1]}\right)d\mu}\\ =&-\frac{V_0^2}{V_1}\end{aligned}$$

where we have used a Hölder inequality to deduce the third line from the second. These two calculations combine to give precisely:

$$\frac{\partial}{\partial t}\left(V_1^2 - V_0V_2\right) \leq 0 \tag{3–3}$$

Furthermore, as the final time is approached we have both V_1 and V_2 tending to zero, by theorem (3–2). Hence $V_1^2 - V_0V_2$ must have been initially non-negative.

This completes the proof of the Aleksandrov-Fenchel inequalities in the case where all the regions are strictly convex and have smooth boundaries. The general case follows by approximation.

Note that in the smooth case, we have equality if and only if $\frac{Q[s]}{Q[s_1]}$ constant at each time. Hence we have $\frac{\partial}{\partial t}s = c(t)s_1$ for every t. Theorem (3–2) therefore gives the result $s(z) = cs_1(z) + \langle z, e\rangle$ for each z, and so D is a scaled translate of D_1.

4. Higher Order Inequalities

In this chapter I will give direct proofs of some of the most interesting inequalities between the mean cross-sectional volumes, using slightly different evolution equations. In particular, I will give proofs of the inequalities:

$$V_k^\ell \geq V_\ell^k V_0^{\ell-k} \qquad 1 \leq k < \ell \leq n.$$

The proof is very similar to the one given in the previous chapter for the 'first order' inequalities. I will use flows with a higher degree of homogeneity to prove these inequalities. The main result is the following:

Theorem 4–1. *Let s_0 be the support function of a smooth, bounded, strictly convex region. Then for $k = 1, \ldots, n$ there exists a unique smooth solution s to the following equation:*

$$\frac{\partial}{\partial t} s = -\left(S_k[s]\right)^{-1} \tag{4–2}$$

where $S_k[s]$ is the k^{th} elementary symmetric function of the principal radii of curvature, given by the following expression:

$$S_k[s] = \mathcal{Q}[\underbrace{s, \ldots, s}_{k \text{ times}}, 1, \ldots, 1].$$

The solution exists for a finite time T, at the end of which it converges to a point. Furthermore, for each $\ell = 1, \ldots, k$ the following quantity is decreasing:

$$V_\ell^{\frac{k+1}{\ell}} - V_{k+1} V_0^{\frac{k+1-\ell}{\ell}} \searrow$$

As in the case treated in the last chapter, this implies:

Corollary 4–3. *The following inequality holds for any ℓ and k with $1 \leq \ell \leq k \leq n$:*

$$V_\ell^{\frac{k+1}{\ell}} - V_{k+1} V_0^{\frac{k+1-\ell}{\ell}} \geq 0.$$

Proof of theorem : The first part of the theorem—the existence of a solution and convergence to a point—follows directly from chapter 1 of section III. It remains to calculate the evolution equations for the mixed volumes. First consider the evolution of V_{k+1}:

$$\begin{aligned}
\frac{\partial}{\partial t} V_{k+1} &= \frac{\partial}{\partial t} \int_{S^n} s S_k[s] \, d\mu \\
&= -\int_{S^n} S_k^{-1} S_k d\mu \\
&\quad - k \int_{S^n} s Q[S_k^{-1}, \underbrace{s, \ldots, s}_{k-1 \text{ times}}, 1, \ldots, 1] d\mu \\
&= -(k+1) \int_{S^n} S_k^{-1} S_k \, d\mu \\
&= -(k+1) V_0.
\end{aligned}$$

Next consider V_ℓ for $\ell \leq k$:

$$\begin{aligned}
\frac{\partial}{\partial t} V_\ell &= \frac{\partial}{\partial t} \int_{S^n} S_\ell[s] d\mu \\
&= -\ell \int_{S^n} Q[S_k^{-1}, \underbrace{s, \ldots, s}_{\ell-1 \text{ times}}, 1, \ldots, 1] d\mu \\
&= -\ell \int_{S^n} \frac{S_{\ell-1}}{S_k} \, d\mu \\
&\leq -\ell \int_{S^n} S_\ell^{-\frac{k+1-\ell}{\ell}} \, d\mu.
\end{aligned}$$

The last line follows from the 'Newton inequalities' $S_\ell^k \geq S_k^\ell$ and $S_{\ell-1}^k \geq S_k^{\ell-1}$. An application of the Hölder inequality then gives:

$$\begin{aligned}
\frac{\partial}{\partial t} V_\ell &\leq -\ell \left(\int_{S^n} S_\ell \, d\mu \right)^{-\frac{k+1-\ell}{\ell}} \left(\int_{S^n} d\mu \right)^{\frac{k+1}{\ell}} \\
&= -\ell V_\ell^{-\frac{k+1-\ell}{\ell}} V_0^{\frac{k+1}{\ell}}.
\end{aligned}$$

Combining these two estimates gives the following:

$$\frac{\partial}{\partial t}\left(V_\ell^{\frac{k+1}{\ell}} - V_{k+1}V_0^{\frac{k+1-\ell}{\ell}}\right) \leq -\frac{k+1}{\ell}V_\ell^{\frac{k+1}{\ell}-1}.\ell V_\ell^{-\frac{k+1-\ell}{\ell}}V_0^{\frac{k+1}{\ell}} + (k+1)V_0.V_0^{\frac{k+1-\ell}{\ell}}$$
$$\leq 0.$$

This completes the proof. ∎

Section V

ENTROPY

INEQUALITIES

1. Decreasing Entropy

In this section I will discuss generalisations of the so-called 'entropy estimates' proved in chapter 5 of section II. In this first chapter I will prove entropy estimates for a wider class of flows, by including a wide class of anisotropic examples. The flows are the same ones as were used in section IV to prove the Aleksandrov-Fenchel inequalities. I will show that the entropy estimates themselves are a consequence of the Aleksandrov-Fenchel inequalities.

In chapter 2, I will extend these entropy estimates to a wider class of equations. These entropy inequalities result in two interesting consequences: First, we are able to investigate expansion flows for curves in the plane, where the speed is homogeneous of degree less than minus one in the curvature. Secondly, I deduce that rescaled solutions do not in general converge to spheres for flows by small positive powers of the Gauss curvature; similar conclusions hold for many other contracting flows with small degree of homogeneity. These are the first examples of non-convergence for parabolic flows of embedded convex hypersurfaces.

Entropy decrease under Aleksandrov-Fenchel flows :

I will consider again the flows used in chapter 3 of section IV to prove the Aleksandrov-Fenchel inequalities. The structure of the equations, which led to the proof in that case, also give beautiful results in the calculations considered here. I will show here that these flows all have entropy estimates analogous to those proved in section II for isotropic examples.

Theorem 1–1. *The scaling invariant quantity*

$$\exp\left\{\frac{1}{V_0}\int_{S^n} s_1 \mathcal{Q}[s_1]\ln\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)d\mu\right\}V_2^{\frac{1}{2}} \tag{1–2}$$

decreases in time under the flow (IV.3–1).

Proof : It is possible to prove this result by imitating the methods used in section II. Instead I will present a simpler proof which uses the Aleksandrov-Fenchel inequalities.

The evolution equation for the quantity (1–2) can be computed as follows: First, from section IV we have:

$$\frac{\partial}{\partial t}V_2 = -2V_0.$$

Next we calculate:

$$\frac{\partial}{\partial t}\int_{S^n} s_1\mathcal{Q}[s_1]\ln\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)d\mu = \int_{S^n} s_1\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\mathcal{Q}\left[s_1\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right]d\mu.$$

Combining these gives the following equation for the evolution of the quantity given above, where we denote $E = \exp\left\{\frac{1}{V_0}\int_{S^n} s_1\mathcal{Q}[s_1]\ln\left(\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right)d\mu\right\}$:

$$\begin{aligned}
\frac{\partial}{\partial t}EV_2^{\frac{1}{2}} &= V_2^{\frac{1}{2}}EV_0^{-1}\int_{S^n} s_1\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\mathcal{Q}\left[s_1\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}\right]d\mu - EV_2^{-\frac{1}{2}}V_0 \\
&= EV_0^{-1}V_2^{-\frac{1}{2}}\left(\int_{S^n} F\mathcal{Q}[F]\,d\mu\int_{S^n} s\mathcal{Q}[s]\,d\mu - \left(\int_{S^n} F\mathcal{Q}[s]\,d\mu\right)^2\right) \\
&\leq 0
\end{aligned}$$

where F is the speed function $s_1\frac{\mathcal{Q}[s_1]}{\mathcal{Q}[s]}$. This follows because the second-last line is exactly the quantity proved to be negative by the Aleksandrov-Fenchel inequalities (compare (IV.2–20)). ∎

Thus the negative time derivative of the entropy quantity is a direct consequence of the Aleksandrov-Fenchel inequalities. In fact the alternative method of proof of the Entropy estimates, given in section II, gives an alternative proof of the Aleksandrov-Fenchel inequalities. This is, however, rather more complicated than the one presented in section IV.

Remark : In the special case of the harmonic mean curvature flow, we know from section I that the solutions become spherical in the limit. It follows that the entropy quantity $EV_2^{\frac{1}{2}}$ is greater than or equal to $|S^n|$ for every compact convex hypersurface, since the value of $EV_2^{\frac{1}{2}}$ on the sphere is exactly $|S^n|$. This is a remarkable inequality which is not at all clear *a priori*. It gives control above and below on the second mean cross-sectional volume V_2 by integrals of S_1:

$$\text{(1–3)} \qquad \exp\left\{\frac{1}{|S^n|}\int_{S^n} \ln S_1 \, d\mu\right\} \leq \left(\frac{V_2}{|S^n|}\right)^{\frac{1}{2}} \leq \frac{1}{|S^n|}\int_{S^n} S_1 \, d\mu.$$

Note that the first and last terms here are the Hölder semi-norms of S_1 with exponent 0 and 1 respectively.

2. New Entropy Flows

In this chapter I will show that there exist scaling-invariant decreasing quantities for many flows. These estimates include previously unknown results for isotropic flows, such as the flows by powers of the Gauss curvature. In these new examples the Aleksandrov-Fenchel inequalities are vital for the proof—the methods of section II do not succeed here. I will give some applications to contraction flows with small degree of homogeneity, proving that these flows do not in general converge to the expected limit shape. Another application of the new estimates gives good control over the shape of curves expanding with speed proportional to the curvature raised to a power less than minus one.

I will consider the following generalisations of the flows considered in the previous chapter—as usual, defined in terms of the support function:

$$\frac{\partial}{\partial t}s = \mathrm{sgn}(\alpha)s_1\left(\frac{\mathcal{Q}[s]}{\mathcal{Q}[s_1]}\right)^{\alpha} \tag{2–1}$$

for $\alpha \neq 0$. The main result is as follows:

Theorem 2–2. *If $\alpha \neq -1$, then the following scaling-invariant quantity decreases under* (2–1)*:*

$$(1+\alpha)\mathrm{sgn}(\alpha)\int_{S^n} s_1\mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s]}{\mathcal{Q}[s_1]}\right)^{1+\alpha} d\mu V_2^{-\frac{1+\alpha}{2}} \searrow$$

Remark : The case $\alpha = -1$ is the case described in the previous chapter.

Proof : First consider the evolution of the integral:

$$\frac{\partial}{\partial t}\int_{S^n} s_1\mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s]}{\mathcal{Q}[s_1]}\right)^{1+\alpha} d\mu = (1+\alpha)\mathrm{sgn}(\alpha)\int_{S^n} s_1\left(\frac{\mathcal{Q}[s]}{\mathcal{Q}[s_1]}\right)^{\alpha}\mathcal{Q}\left[\frac{\partial}{\partial t}s\right] d\mu.$$

The evolution of the mixed volume V_2 is given by:

$$\frac{\partial}{\partial t}V_2 = -2\mathrm{sgn}(\alpha)\int_{S^n} s_1\left(\frac{\mathcal{Q}[s]}{\mathcal{Q}[s_1]}\right)^{\alpha}\mathcal{Q}[s]\,d\mu.$$

Combining these gives the evolution equation for the whole quantity:

$$\begin{aligned}\frac{\partial}{\partial t}V_2^{-\frac{1+\alpha}{2}}\int_{S^n} s_1\mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s]}{\mathcal{Q}[s_1]}\right)^{1+\alpha} d\mu &= (1+\alpha)\mathrm{sgn}(\alpha)V_2^{-\frac{1+\alpha}{2}}\int_{S^n} F\mathcal{Q}\left[F\right]\,d\mu\\
&\quad-(1+\alpha)\mathrm{sgn}(\alpha)V_2^{-\frac{3+\alpha}{2}}\left(\int_{S^n} F\mathcal{Q}[s]\,d\mu\right)^2\\
&=(1+\alpha)\mathrm{sgn}(\alpha)V_2^{-\frac{1+\alpha}{2}}\\
&\quad\times\left(\int_{S^n} F\mathcal{Q}[F]\,d\mu-\frac{\left(\int_{S^n} F\mathcal{Q}[s]\,d\mu\right)^2}{\int_{S^n} s\mathcal{Q}[s]\,d\mu}\right)\end{aligned}$$

where $F = s_1\left(\frac{\mathcal{Q}[s]}{\mathcal{Q}[s_1]}\right)^{\alpha}$. The term in brackets here is precisely an Aleksandrov-Fenchel difference, and the result follows. ∎

Let us consider some of the consequences of this: We can divide the flows into cases depending on their homogeneity. In the case of expansion flows, we have $\alpha > 0$, and so the quantity

$$V_2^{-\frac{1+\alpha}{2}}\int_{S^n} s_1\mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s]}{\mathcal{Q}[s_1]}\right)^{1+\alpha} d\mu$$

is decreasing. One can see that this is a good estimate, because an application of the Hölder inequality gives the following:

$$\begin{aligned}V_2^{-\frac{1+\alpha}{2}}\int_{S_n} s_1\mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s]}{\mathcal{Q}[s_1]}\right)^{1+\alpha} d\mu &\geq \frac{\left(\int_{S_n} s_1\mathcal{Q}[s_1]\left(\frac{\mathcal{Q}[s]}{\mathcal{Q}[s_1]}\right)d\mu\right)^{1+\alpha}}{V_0^{\alpha}V_2^{\frac{1+\alpha}{2}}}\\
&=V_0^{-\alpha}\left(\frac{V_1^2}{V_2}\right)^{\frac{1+\alpha}{2}}.\end{aligned}$$

Thus the decreasing scaling-invariant quantity gives a bound on the isoperimetric ratio $\frac{V_1^2}{V_2}$. We have already seen in chapter 3 of section III that expansion flows with $\alpha \leq 1$ behave extremely well; this estimate gives some hope that flows with higher homogeneity can also be handled; in the case of expanding convex curves, one can easily see that this estimate is sufficient to give convergence:

Theorem 2–3. *Let $\alpha > 1$. Suppose $n = 2$, and consider the evolution equation (2–1). Let s_0 be the support function of a strictly convex, embedded initial curve, and assume that this curve encloses the origin. There exists a unique smooth solution to (2–1) on a finite time interval $[0, T)$. After rescaling about the origin to give constant enclosed area, the hypersurfaces converge smoothly to the limit shape s_1.*

Remark : Note that the form of the equations is not at all restrictive in the case of curves.

Proof: The short-time existence and uniqueness of the solution follows directly from the strict parabolicity of the equations.

Note that the finiteness of the interval of existence of the solution follows immediately by considering the evolution of an enclosed copy of s_1. This expands with radius given by an expression of the form $(C - (\alpha - 1)t)^{-\frac{1}{\alpha-1}}$, and consequently becomes infinite in finite time. The parabolic maximum principle implies that a solution which is initially enclosed remains enclosed as long as the solution exists.

The estimate (2–2) above gives a bound on the isoperimetric ratio of the solution. The Bonnesen inequality therefore implies the following (see [**BZ**], page 3 and page 148; also chapter 2 of section III):

$$\frac{\rho_+}{\rho_-} \leq C$$

where ρ_+ and ρ_- are the isotropic analogues of the inradius and circumradius (see chapter 5 of section I) defined by:

$$\rho_- = \sup\{\rho : \rho s_1 + \langle z, p\rangle \leq s \text{ for some } p \text{ in } \mathbb{R}^{n+1}.\}$$

$$\rho_+ = \inf\{\rho : \rho s_1 + \langle z, p\rangle \geq s \text{ for some } p \text{ in } \mathbb{R}^{n+1}.\}.$$

We can show that ρ_- approaches infinity towards the final time: This is similar to the analysis in chapter 7 of section I. First note that the speed Φ is increasing, so we have a bound below on the radius of curvature; as long as the circumradius remains bounded by some constant R, a consideration of the evolution equation for $\frac{\Phi}{R-s}$ shows that Φ remains bounded. This gives bounds above and below on the curvature, and higher estimates follow by standard theory. In fact, given the bound on the ratio of ρ_- and ρ_+ proved above, this is enough to give a bound above on Φ on the rescaled hypersurfaces. The evolution equation for $\frac{\Phi}{s+R}$ for $R > 0$ gives bounds below on Φ on the rescaled hypersurfaces. The Harnack inequality of section II gives Hölder estimates on the curvature, and higher regularity follows by Schauder theory.

This proves that there is a subsequence of times for which the rescaled hypersurfaces converge in C^∞ to a smooth convex hypersurface. The fact that this hypersurface has support function s_1 now follows because the quantity given in (2–2) is strictly decreasing otherwise. Strong convergence follows as in chapter 7 of section II. ∎

Next we consider the case $\alpha < -1$. In this case, the quantity

$$\int_{S^n} s_1 Q[s_1] \left(\frac{Q[s_1]}{Q[s]}\right)^{|1+\alpha|} d\mu V_2^{\frac{|1+\alpha|}{2}}$$

is decreasing. This may be compared with the estimate for $\alpha = -1$:

$$\frac{1}{V_0}\int_{S^n} s_1 Q[s_1]\left(\frac{Q[s_1]}{Q[s]}\right)^{|1+\alpha|} d\mu\, V_2^{\frac{|1+\alpha|}{2}}$$
$$\geq \left(V_2^{\frac{1}{2}} \exp\left\{\frac{1}{V_0}\int_{S^n} s_1 Q[s_1]\ln\left(\frac{Q[s_1]}{Q[s]}\right) d\mu\right\}\right)^{|1+\alpha|}.$$

Finally, we consider the case $-1 < \alpha < 0$. In this case we have an increasing quantity:

$$V_2^{-\frac{1+\alpha}{2}}\int_{S^n} s_1 Q[s_1]\left(\frac{Q[s]}{Q[s_1]}\right)^{1+\alpha} d\mu \nearrow$$

This has the consequence that the solutions of these flows (which, by section III, converge to points) do not in general converge to the limit s_1 after rescaling. To see this, note that the Aleksandrov-Fenchel inequality gives the following:

$$\int_{S^n} s_1 Q[s_1]\left(\frac{Q[s]}{Q[s_1]}\right) d\mu \geq V_2^{\frac{1}{2}} V_0^{\frac{1}{2}}$$

with strict inequality unless $s = cs_1 + \langle z, p\rangle$. If s is smooth, then $\frac{Q[s]}{Q[s_1]}$ is bounded above and below. It follows that for any smooth hypersurface not congruent to s_1, there is a sufficiently small $\varepsilon > 0$ such that

$$\left(\int_{S^n} s_1 Q[s_1]\left(\frac{Q[s]}{Q[s_1]}\right)^{1-\epsilon} d\mu\right)^{\frac{1}{1-\epsilon}} > V_2^{\frac{1}{2}} V_0^{\frac{1+\epsilon}{2-2\epsilon}}.$$

It follows that the flows (2–1) with exponent α in the range $-\varepsilon < \alpha < 0$ do not converge with initial condition s. It is not clear, in general, whether there are counterexamples for every α between 0 and -1. However, if we take s_1 to be the support function of a unit square in the plane, and assume that s is the support function of a rectangle, then the contraction flows with degree of homogeneity between 0 and -1 fail terribly: Although all such rectangles contract to points, they actually become less square during the evolution; The only initial condition which has a square limit is the square itself!

Remark : I have presented here only the generalisations of the flows (IV.3–1). In fact, obvious generalisations of these estimates work also for the higher order flows (IV.4–2). These include flows by the powers of the Gauss curvature. The same calculations as above show that expanding flows by powers of the Gauss curvature (with any positive homogeneity in the principal radii of curvature) give solutions which become spherical in the limit. Similarly, we can deduce that contraction flows by small powers of the Gauss curvature do not converge to spheres after rescaling. These results hold in any dimension.

Section VI

CONTRACTING CONVEX HYPERSURFACES IN RIEMANNIAN SPACES

1. Introduction

In the previous sections, we have considered many flows of hypersurfaces in Euclidean space. In this last section we consider the analogous flows for hypersurfaces in Riemannian spaces. We will concentrate on adapting the flows and techniques of section I to this more complicated situation. We prove that any compact hypersurface satisfying a sharp convexity condition is necessarily the boundary of an immersed disc (Theorem (1–5).

Let M^n be a smooth, connected compact manifold of dimension $n \geq 2$ without boundary, and let (N^{n+1}, g^N) be a complete smooth Riemannian manifold satisfying the following conditions:

$$\begin{aligned} -K_1 \leq \sigma^N \leq K_2 \\ |\nabla^N R^N|_{g^N} \leq L \end{aligned} \tag{1–1}$$

for some non-negative constants K_1, K_2 and L. Here σ^N is any sectional curvature of N^{n+1}, ∇^N is the Levi-Civita connection corresponding to g^N, and R^N is the Riemann tensor on N^{n+1}.

Suppose $\varphi_0 : M^n \to N^{n+1}$ is a smooth immersion of M^n. We seek a solution $\varphi : M^n \times [0, T) \to N^{n+1}$ to an equation of the following form:

$$\begin{aligned} \frac{\partial}{\partial t}\varphi(x,t) &= -f(\lambda(\mathcal{W}(x,t)))\nu(x,t) \\ \varphi(x,0) &= \varphi_0(x) \end{aligned} \tag{1–2}$$

where $\nu(x,t)$ is a unit normal to $\varphi_t(M)$ at $\varphi_t(x)$ in TN^{n+1}, $\mathcal{W}(x,t)$ is the Weingarten map on TM^n induced by φ_t, λ is the map from $T^*M^n \otimes TM^n$ to R^n which gives the eigenvalues of a map, and f is a smooth symmetric function. Several further conditions are required of the function f; these are given in chapter 3.

Huisken [**Hu2**] has considered the mean curvature flow in this setting; in this case $f(\lambda) = \sum_{i=1}^{n} \lambda_i$. The main theorem of [**Hu2**] may be stated as follows:

Theorem 1–3. *Suppose M^n, N^{n+1}, and φ_0 are as above, and assume in addition that the injectivity radii $i_y(N)$ of N^{n+1} have a positive lower bound $i(N)$, and that the principal curvatures of φ_0 satisfy the inequality:*

$$H\lambda - nK_1 > \frac{n^2 L}{H} \tag{1–4}$$

where $H = f(\lambda_1, ..., \lambda_n) = \sum_{i=1}^{n} \lambda_i$. Then there exists a unique smooth solution to (1–2) on a maximal time interval $[0, T)$. The immersions φ_t converge uniformly to a constant $p \in N^{n+1}$ as t approaches T. The rescaled immersions $\tilde{\varphi}_\tau$ obtained by rescaling a neighbourhood of p by a factor $(2n(T-t))^{-\frac{1}{2}}$ converge to the unit sphere $S_1^n(0)$ in Euclidean space, exponentially in C^∞ with respect to the natural time parameter $\tau = -\frac{1}{2}\ln(1 - \frac{t}{T})$.

The details of the rescaling process will be explained in chapter 6. This theorem gives optimal results in the case of a locally symmetric background space; the particular case of hypersurfaces of the sphere was developed further in [**Hu3**]. In more general spaces, the appearance of the derivatives of R^N in (1–4) is undesirable.

Here we consider a class of fully non-linear flow equations which does not include the mean curvature flow. The structure of the equations is similar in many respects to the mean curvature flow, and to the class of equations considered in section I. A typical example is the flow by shifted harmonic mean curvature, for which $f(\lambda) = \left(\frac{1}{n}\sum_{i=1}^{n}(\lambda_i - \sqrt{K_1})^{-1}\right)^{-1}$. The main result achieved here is the following:

Theorem 1–5. *Let M^n and N^{n+1} be as above. Assume that f satisfies the conditions (3–1), and every principal curvature λ of φ_0 satisfies the following condition:*

$$\lambda > \sqrt{K_1}. \tag{1-6}$$

Then there exists a unique smooth solution to (1–2) on a maximal time interval $[0, T)$, and the immersions φ_t converge uniformly to a constant p in N^{n+1} as t approaches T. Expanding a neighbourhood of p by a factor $(2(T-t))^{-\frac{1}{2}}$ gives rescaled immersions $\tilde{\varphi}_\tau$ which converge in C^∞ to the unit sphere about the origin in Euclidean space, exponentially with respect to the natural rescaled time parameter $\tau = -\frac{1}{2}\ln(1-\frac{t}{T})$.

The hypotheses of this theorem differ from those in (1–3) in two important respects: No lower bound on the injectivity radius of N is required, and (1–4) is replaced by (1–6). For locally symmetric background spaces ($L = 0$), the new condition is slightly more restrictive than (1–4), but still sharp in the sense that there are counterexamples which satisfy (1–6) with equality. Furthermore, the removal of the dependence on L is a significant improvement in the general case, allowing some useful geometric applications which will be discussed in section 7. Note that the condition (1–6) is just enough to ensure that the hypersurface has non-negative sectional curvatures.

Corollary 1–7. *Any compact hypersurface in N with principal curvatures greater than $\sqrt{K_1}$ is diffeomorhic to a sphere, and bounds an immersed disc.*

The organisation of this section is as follows: Chapter 2 introduces the new notation for the Riemannian case, and gives some useful preliminary results. Chapter 3 contains details of the evolution equations—the form of the function f, the equivalence of the system (1–2) locally to a scalar equation, short-time existence and uniqueness of solutions, and the induced evolution equations for some geometric quantities. Chapter 4 deals with the preservation of convexity and the pinching of principal curvatures; this requires only minor modifications from the proof for the Euclidean case in section I. The application of these estimates, however, is more difficult than in the Euclidean case—the quantities dealt with there can no longer be defined, and one must use more local estimates. These are developed in chapter 5: The local graphical parametrisation of the flow, developed in chapter 3, is used to prove local Hölder estimates on the curvature of the immersions. This is accomplished using results from Krylov [**K**]. These estimates allow us to prove, in chapter 6, the convergence of a subsequence of appropriately rescaled hypersurfaces to a strictly convex pinched hypersurface in Euclidean space. A recent result of Hamilton [**Ha6**] implies that this limit hypersurface is compact, and the proof of convergence to a point follows directly. The convergence of the rescaled immersions to a sphere follows using techniques similar to those in the analogous part of section I. Chapter 7 concludes with an extension to slightly different flow equations, an application of the main theorem to give a new proof of the 1/4-pinching sphere theorem, and a generalisation of this proof to give a new "dented sphere" theorem.

2. Notation and Preliminary Results

As far as possible the notation employed in this section is consistent with that in section I.

The background space N^{n+1} is supplied with a metric g^N, and corresponding connection ∇^N and Riemann tensor R^N. Each immersion φ_t of M^n induces a metric g, a connection ∇, and a Riemann curvature tensor R on the tangent bundle TM^n:

$$
\begin{aligned}
&g(u,v) = g^N\left(T\varphi(u), T\varphi(v)\right)\\
&\nabla_u v = T_x\varphi^{-1}\left(\pi_x\left(\nabla^N_{T\varphi(u)}T\varphi(v)\right)\right)\\
&R(u,v,w) = \left(\nabla_v\nabla_u - \nabla_u\nabla_v - \nabla_{[u,v]}\right)w\\
&R(u,v,w,z) = g^N\left(R(u,v,w),z\right)
\end{aligned}
\tag{2–1}
$$

for all u, v, w and z in T_xM^n. Here $T_x\varphi$ is the derivative of φ, and π_x is the projection of $T_{\varphi(x)}N^{n+1}$ onto the image of $T_x\varphi$. For a nonzero simple 2-plane $X = u\wedge v$, the sectional curvature $\sigma^N(X)$ is given by $\frac{R(u,v,u,v)}{|u\wedge v|^2}$. It is convenient to use the Riemann tensor to define a map $\mathcal{R} : T^*M\otimes TM \to T^*M\otimes TM$ generated by the equation $\mathcal{R}(g^*(u\otimes v))(w) = R(u,w,v)$. Note that $\mathcal{R}$ sends symmetric maps to symmetric maps.

For any point y in N, the exponential map $\exp_y : T_yN \to N$ can be defined: For a vector u in T_yN, $\exp_y(u)$ is the endpoint of the geodesic from y which has tangent in the direction of u at y, and length equal to the length of u. This is always a diffeomorphism on a small neighbourhood of the origin in T_yN. The injectivity radius $i_y(N)$ is the least upper bound of the set of r for which the exponential map is a diffeomorphism on the ball of radius r about the origin in T_yN.

There are several different metrics which will be used in the course of the proof. The norm on tensor bundles associated with a metric g will be denoted by $|.|_g$.

A convenient notation is the following: for a tensor $\mathcal{T}$ in T^*N, we write $\mathcal{T}(u)$ in place of $\mathcal{T}(T\varphi(u))$, for any vector field u in TM. This generalises in an obvious way to higher tensors.

In analogy with the Euclidean case, the normal component of the connection on N^{n+1} gives the second fundamental form $\mathit{II} \in T^*M \otimes T^*M$, which is symmetric with respect to the metric g:

$$\mathit{II}(u,v) = -g^N\left(\nabla^N_{T\varphi(u)} T\varphi(v), \nu\right) \tag{2–2}$$

for all u and v in T_xM^n. The Codazzi and Gauss equations are slightly different from the Euclidean case:

(2–3)

$$\nabla \mathit{II}(u,v,w) = \nabla \mathit{II}(v,u,w) + R^N(\nu,u,v,w)$$

(2–4)

$$R(u,v,w,z) = \mathit{II}(u,w)\mathit{II}(v,z) - \mathit{II}(v,w)\mathit{II}(u,z) + R^N(u,v,w,z)$$

for all u, v,w and z in T_xM^n.

The Weingarten map $\mathcal{W} : TM^n \to TM^n$ gives the change of the normal with respect to the ambient connection:

$$\mathcal{W}(u) = T\varphi^{-1}\left(\nabla^N_{T\varphi(u)}\nu\right) \tag{2–5}$$

for all u in T_xM^n. As in the Euclidean case the Weingarten relation relates the second fundamental form to the Weingarten map:

$$\mathit{II}(u,v) = g\left(\mathcal{W}(u), v\right). \tag{2–6}$$

A hypersurface is called convex if the Weingarten map is positive everywhere. We will also refer to a hypersurface as α-convex if the Weingarten map has all eigenvalues greater than α. This notation should not be confused with that in section II, which will not be employed here.

A useful identity involving the second derivatives of the second fundamental form is Simons's identity. This combines the Codazzi equation (2–3), the formula for interchange of derivatives in terms of curvature which derives from (2–1), and the Gauss equation (2–4) for the Riemann tensor:

(2–7)

$$\begin{aligned}
\mathrm{Hess}_{\nabla}\mathit{II}(u,v,w,z) =& \mathrm{Hess}_{\nabla}\mathit{II}(w,z,u,v) + \mathit{II}(u,v)\mathit{II}^2(w,z) - \mathit{II}(w,z)\mathit{II}^2(u,v) \\
&+ \mathit{II}(u,z)\mathit{II}^2(w,v) - \mathit{II}(w,v)\mathit{II}^2(u,z) \\
&+ R^N(u,w,v,\mathcal{W}(z)) - R^N(w,u,z,\mathcal{W}(v)) \\
&+ R^N(u,z,v,\mathcal{W}(w)) - R^N(w,v,z,\mathcal{W}(u)) \\
&+ \mathit{II}(u,v)R^N(w,\nu,z,\nu) - \mathit{II}(w,z)R^N(u,\nu,v,\nu) \\
&+ \nabla^N R^N(u,v,w,z,\nu) - \nabla^N R^N(w,z,u,v,\nu)
\end{aligned}$$

for all vectors u, v, w, and z in TM.

In chapter 3 we will make use of special local coordinates on N^{n+1} which are particularly convenient for the local graphical parametrisation of the evolution equations (see (3–2)).

Suppose $\psi_0 : \Sigma^n \to N^{n+1}$ is a smooth immersion of a compact manifold Σ (possibly with a smooth boundary). We wish to extend ψ to $\Sigma^n \times (-\epsilon, \epsilon)$ by the following equations:

$$\frac{\partial}{\partial s}\psi(\xi, s) = \hat{\nu}(\xi, s) \qquad \text{(2–8)}$$
$$\psi(\xi, 0) = \psi_0(\xi)$$

for every ξ in Σ^n and every s in $(-\epsilon, \epsilon)$, where $\hat{\nu}(\xi, s)$ is a unit normal to $\psi(\Sigma^n, s)$ at $\psi(\xi, s)$. Where the maps $\psi^{(s)} = \psi(., s)$ are nondegenerate, the corresponding induced metric, connection and second fundamental form on Σ are denoted by $g^{(s)}$, $\nabla^{(s)}$, and $I\!I^{(s)}$. The map ψ is called a *graphical coordinate system over* ψ_0.

Lemma 2–9. *For Σ and ψ_0 as above, there exists a map $\psi : \Sigma^n \times (-\epsilon, \epsilon)$ satisfying* (2–8), *for some sufficiently small positive ϵ. There exists a constant C depending on ψ_0 and N such that:*

$$C^{-1}g^{(0)}(u,u) \le g^{(s)}(u,u) \le Cg^{(0)}(u,u) \qquad \text{(2–10)}$$
$$|I\!I^{(s)}(u,u)|_{g^{(0)}} \le C$$
$$|\nabla^{(s)}_u v - \nabla^{(0)}_u v|_{g^{(0)}} \le C$$

for all u in $T\Sigma^n$.

Proof : This follows from the induced variation equations for geometric quantities, which are given by (3–15), substituting 1 for f. ∎

There is a special case of such graphical coordinates which is very important for proving local estimates: Let y_0 be a point in N, P an n-dimensional subspace of $T_{y_0}N$, and e_0 a unit normal to P in $T_{y_0}N$. Define a map $\psi_0 : P \to N$ according to the equation:

$$\psi_0(\xi) = \exp_{y_0}(\xi) \qquad \text{(2–11)}$$

for every ξ in P. On a region Σ of P where ψ_0 is nondegenerate, it can be used as the initial immersion in equation (2–8), where we use the unit normal given by

$$\hat{\nu}(\xi, 0) = \left(T_\xi \exp_{y_0}\right)(e_0). \tag{2–12}$$

The map ψ produced in this way is called the *graphical coordinate system over* P.

The metric on $P \subset T_{y_0}N$ will be denoted by $\langle .,. \rangle$, and the corresponding norm by $|.|$. The standard (flat) connection on P is denoted by d.

Lemma 2–13. *Suppose N satisfies (1–1) with $K_1 = K_2 = L = 1$. Then the graphical coordinate system ψ over any n-dimensional hyperplane P is nondegenerate on the domain $B_{\rho_0} \times (-\rho_0, \rho_0) \subset P \oplus Re_0$ for some fixed $\rho_0 > 0$ depending only on n, where $\Sigma = B_{\rho_0}$ is the ball of radius ρ_0 in P. The following estimates hold for some fixed constant C:*

$$\begin{aligned} C^{-1}|u|^2 \le g^{(s)}(u,u) &\le C|u|^2 \\ |I\!I^{(s)}(u,u)| &\le C \\ |\nabla_u^{(s)} v - d_u v| &\le C \\ |\nabla^{(s)} I\!I^{(s)}| &\le C \end{aligned} \tag{2–14}$$

for all u and v in P.

Proof : The assumptions (1–1) give uniform control over the curvature of N and its derivative. This allows control over the Hamilton-Jacobi equations (2–11) and (2–8) which define ψ, and the induced variation equations for the metric and curvature ∎

A hypersurface can be described locally using the graphical coordinates given by (2–9). For a smooth function $s : \Sigma^n \to (-\epsilon, \epsilon)$, define an immersion $\varphi : \Sigma^n \to N$

by

(2–15) $$\varphi(\xi) = \psi(\xi, s(\xi))$$

for all ξ in Σ^n. For such a graph we can calculate the metric, curvature, and connection of the immersion:

(2–16)
$$\nu = \frac{\hat{\nu}^{(s)} - \nabla^{(s)} s}{\sqrt{1 + |\nabla s|^2_{g^{(s)}}}}$$

(2–17)
$$g = g^{(s)} + \nabla s \otimes \nabla s$$

(2–18)
$$\nabla_u v - \nabla^{(s)}_u v = \frac{\mathrm{Hess}_{\nabla^{(s)}} s(u,v)}{1 + |\nabla s|^2_{g^{(s)}}} \nabla^{(s)} s$$

(2–19)
$$I\!I(u,v) = \frac{\left(I\!I^{(s)}(u,v) + I\!I^{(s)}(v, \nabla^{(s)} s)\nabla_u s + I\!I^{(s)}(u, \nabla^{(s)} s)\nabla_v s - \mathrm{Hess}_{\nabla^{(s)}} s(u,v)\right)}{\sqrt{1 + |\nabla s|^2_{g^{(s)}}}}$$

for all vectors u and v in $T_\xi P \cong P$.

3. The Evolution Equations

The speed functions f must satisfy conditions similar to those required in chapter 3 of section I:

Conditions 3–1.

(1). f is a symmetric function which is smooth on $\Gamma_\alpha = \{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i \geq \alpha\}$, and continuous on $\bar{\Gamma}_\alpha$, where $\alpha = \sqrt{K_1}$.

(2). f is strictly increasing in each argument: $\frac{\partial f}{\partial \lambda_i} > 0$ for $i = 1, \ldots, n$ at every point of Γ_α.

(3). f is homogeneous of degree one in $(\lambda_1 - \alpha, \ldots, \lambda_n - \alpha)$.

(4). f is strictly positive on Γ_α, and $f(1, \ldots, 1) = 1$.

(5). f is concave on Γ_α.

(6). $f = 0$ on $\partial\Gamma_\alpha$.

(7). $\sup_{\lambda \in \Gamma_\alpha} |Df| < \infty$.

For convenience the composition $f \circ \lambda$ will be denoted by F, and its derivatives by $\dot{F}$, $\ddot{F}$, etc., as in section I. Note that the shifted harmonic mean curvature, given by $f = \left(\frac{1}{n} \sum_{i=1}^{n} (\lambda_i - \alpha)^{-1}\right)^{-1}$ satisfies all of the conditions (3–1).

Condition (2) ensures that equation (1–2) is a degenerate parabolic system of partial differential equations. The second part of condition (4) is only a normalisation condition, and can always be satisfied by rescaling time. Note that condition (6) rules out the mean curvature flow, and condition (7) rules out the flow by the nth root of the Gauss curvature. In the case where N has non-negative sectional curvatures, the allowed flows are a subset of the allowed flows in the Euclidean case. More generally, we require the more complicated homogeneity condition (3) in order to overcome negative curvature of the background space.

The proof of short-time existence and uniqueness of solutions is essentially the same as in section I, but the graphical parametrisation is somewhat more complicated because of the background geometry. Some results concerning the graphical parametrisation of the flow are necessary.

Lemma 3–2. *Let $\psi : \Sigma^n \times (-\epsilon, \epsilon) \to N^{n+1}$ be a nondegenerate map given by (2–9), and $\varphi_0 : M^n \to N$ a smooth α-convex immersion. Suppose there exists a nondegenerate map $\chi_0 : \Sigma^n \to M^n$, and a smooth function $s_0 : \Sigma^n \to (-\epsilon, \epsilon)$ such that*

$$\varphi_0(\chi_0(\xi)) = \psi(\xi, s_0(\xi)) \tag{3–3}$$

$$g^N\left(\nu(\chi_0(\xi)), \hat{\nu}^{(s_0)}(\xi)\right) > 0 \tag{3–4}$$

for all ξ in Σ^n. If $\varphi : M^n \times [0,T) \to N$ is a family of α-convex immersions satisfying (1–2), then for sufficiently small $t_0 > 0$ there exists a smooth family of non-degenerate maps $\chi : \Sigma^n \times [0,t_0) \to M^n$ and a smooth family of functions $s : \Sigma^n \times [0,t_0) \to (-\epsilon, \epsilon)$ such that

$$\varphi_t(\chi_t(\xi)) = \psi(\xi, s_t(\xi)) \tag{3–5}$$

for all (ξ,t) *in* $\Sigma^n \times [0,t_0)$*. Furthermore,* s *satisfies the inequality*

(3–6)
$$\frac{\left(\mathit{II}^{(s)}(u,u)+2\mathit{II}^{(s)}(u,\nabla^{(s)}s)\nabla_u s-\mathrm{Hess}_{\nabla^{(s)}}s(u,u)\right)}{\sqrt{1+|\nabla s|^2_{g^{(s)}}}} \geq \alpha\left(|u|^2_{g^{(s)}}+(\nabla_u s)^2\right)$$

for all (ξ,t) *in* $\Sigma^n \times [0,t_0)$ *and all* u *in* $T_\xi\Sigma^n$*. The following strictly parabolic equation holds on* $\Sigma\times[0,t_0)$*:*

$$\frac{\partial}{\partial t}s(\xi,t)=f\circ\lambda(((g^{(s)})^*g)^{-1}\circ\mathcal{A}) \tag{3–7}$$
$$s(\xi,0)=s_0(\xi)$$

where g *is given in terms of* s *by* (2–17)*, and* $\mathcal{A}$ *is the map given by*

$$\mathcal{A}=(g^{(s)})^*\left(\mathrm{Hess}_{\nabla^{(s)}}s-\mathit{II}^{(s)}-\nabla s\otimes(\mathcal{W}^{(s)})^\dagger(\nabla s)-(\mathcal{W}^{(s)})^\dagger(\nabla s)\otimes\nabla s\right); \tag{3–8}$$

Here $(\mathcal{W}^{(s)})^\dagger$ *is the adjoint of* $\mathcal{W}^{(s)}$*.*

Conversely, if $s:\Sigma^n\times[0,t_0)\to(-\epsilon,\epsilon)$ *is smooth and satisfies* (3–6) *and* (3–7)*, then for every point* (ξ_1,t_1) *in* $\Sigma^n\times[0,t_0)$ *there exists a manifold* $\bar{M}$ *and a smooth family of diffeomorphisms* $\bar\chi$ *of* $\bar M\times[t_1,t_2)$ *onto regions of* Σ^n *containing* ξ_1*, for some* $t_2\in(t_1,t_0)$*, such that the map* $\bar\varphi:\bar M\times[t_1,t_2)\to N$ *given by*

$$\bar\varphi_t(\bar x)=\psi(\bar\chi_t(\bar x),s_t(\bar\chi_t(\bar x))) \tag{3–9}$$

is a smooth family of α*-convex immersions satisfying* (1–2)*. If* s *is produced from* φ *as above, then there exists a nondegenerate map* $\phi:\bar M\to M$ *such that*

$$\varphi_t(\phi(\bar x))=\bar\varphi_t(\bar x) \tag{3–10}$$

for all $(\bar x,t)$ *in* $\bar M\times[0,t_0)$*.*

Proof : Let φ be a solution to (1–2) as above, and suppose s_0 and χ_0 give φ_0

by equation (3–3). Consider the ordinary differential equations

(3–11)
$$\frac{d}{dt}\chi_t(\xi) = -\frac{F(\chi_t(\xi))}{g^N(\nu(\chi_t(\xi)), \hat{\nu}^{(s_t)}(\xi))}(T_{\chi_t(\xi)}\varphi_t)^{-1}(\pi_{\chi_t(\xi)}\hat{\nu}^{(s_t)}(\xi))$$
$$\frac{d}{dt}s_t(\xi) = -\frac{F(\chi_t(\xi))}{g^N(\nu(\chi_t(\xi)), \hat{\nu}^{(s_t)}(\xi))}.$$

There exist solutions χ and s to (3–11) on a time interval $[0, t_0)$, with $|s_t(\xi)| < \epsilon$. The consistency of the equations is guaranteed by the following calculation:

$$\begin{aligned}\frac{\partial}{\partial t}\varphi_t(\chi_t(\xi))) &= -F(\chi_t(\xi))\nu(\chi_t(\xi)) - \frac{F(\chi_t(\xi))}{g^N(\nu(\chi_t(\xi)), \hat{\nu}^{(s_t)}(\xi))}\pi_{\chi_t(\xi)}\hat{\nu}^{(s_t)}(\xi)\\ &= -\frac{F(\chi_t(\xi))}{g^N(\nu(\chi_t(\xi)), \hat{\nu}^{(s_t)}(\xi))}\hat{\nu}^{(s_t)}(\xi)\\ &= \frac{\partial}{\partial t}\psi(\xi, s_t(\xi))\end{aligned}$$

for all ξ and t. Hence equation (3–5) holds on the interval $[0, t_0)$. The equations (3–6) and (3–7) follow immediately from the expressions (2–19) and (2–17): The first since φ is α-convex, and the second from (1–2).

Now consider the converse situation: Suppose $s : \Sigma^n \times [0, t_0) \to (-\epsilon, \epsilon)$ is a solution to (3–7), and (ξ_1, t_1) is in $\Sigma^n \times [0, t_0)$. Let $\bar{M}^n$ be a small open neighbourhood of ξ_1 in Σ^n, and define $\bar{\chi}_{t_1} : \bar{M}^n \to \Sigma^n \times [0, t_0)$ by $\bar{\chi}_{t_1} = \mathrm{Id} \times \{t_1\}$. Extend $\bar{\chi}$ to a region $\bar{M}^n \times [t_1, t_2)$ (taking $t_2 - t_1$ and $\bar{M}^n$ sufficiently small) by the following differential equation:

(3–12)
$$\frac{\partial}{\partial t}\bar{\chi}_t(\bar{x}) = -\left(\frac{1}{1 + |\nabla s|^2_{g(s)}}\frac{\partial}{\partial t}s\right)\nabla^{(s)}s$$

where the right hand side is evaluated at $\bar{\chi}_t(\bar{x})$, for all $(\bar{x}, t)$ in $\bar{M}^n \times [t_1, t_2)$. The definition (3–9) of $\bar{\varphi}$ then gives:

$$\begin{aligned}
\frac{\partial}{\partial t}\bar{\varphi}(\bar{x}) =& \left(\frac{\partial}{\partial t}s\right)\left(1-\frac{|\nabla s|^2}{1+|\nabla s|^2}\right)\hat{\nu}^{(s_t)}(\bar{\chi}_t(\bar{x})) \\
&-\left(\frac{\partial}{\partial t}s\right)\frac{T\psi(\nabla^{(s)}s)}{1+|\nabla s|^2} \\
=&\frac{1}{\sqrt{1+|\nabla s|^2}}\left(\frac{\partial}{\partial t}s\right)\nu(\bar{x}) \\
=&-F(\bar{\varphi}_t(\bar{x}))\nu(\bar{x})
\end{aligned}$$

by equation (2–16), and $\bar{\varphi}$ satisfies (1–2). Finally, if s is produced from a solution φ, define $\psi : \bar{M}^n \to M^n$ by

$$\psi(\bar{x}) = \chi_t \circ \bar{\chi}_t(\bar{x}) \tag{3–13}$$

which is well-defined since $\chi_t \circ \bar{\chi}_t$ satisfies the equation $\frac{\partial}{\partial t}\chi_t \circ \bar{\chi}_t = 0$. ∎

Now consider the case where $\Sigma^n = M^n$ and $\psi_0 = \varphi_0$. The following result is easily obtained from (3–2):

Theorem 3–14. *There exists a unique smooth solution to equation* (1–2) *on some time interval* $[0, T)$.

Proof: There exists a solution for a short time to the equation (3–7) with zero initial conditions, since it is strictly parabolic. This gives a solution to (1–2) by the lemma above, satisfying the correct initial conditions.

Suppose there are two solutions φ^1 and φ^2 to (1–2) with the same initial condition φ_0. This gives two solutions to (3–7) with the same initial conditions, which are therefore identical. It follows that φ^1 and φ^2 are identical up to a time-independent diffeomorphism, and therefore identical since they have the same initial condition. ∎

The evolution equations satisfied by the metric, normal, and curvature of the immersions φ_t of a solution to (1–2) are similar to the Euclidean case:

Theorem 3–15.

$$\frac{\partial}{\partial t} g = -2F\mathit{II} \tag{3–16}$$

$$\frac{\partial}{\partial t}\nu = T\varphi(\nabla F) \tag{3–17}$$

$$\frac{\partial}{\partial t}\mathit{II} = \mathrm{Hess}_{\nabla} F - F\mathit{II}^2 + FR^N(.,\nu,.,\nu) \tag{3–18}$$

$$\frac{\partial}{\partial t}\mathcal{W} = g^*\mathrm{Hess}_{\nabla} F + F\mathcal{W}^2 + F\mathcal{R}^N(g^N)^*(\nu\otimes\nu) \tag{3–19}$$

$$\frac{\partial}{\partial t}F = \mathcal{L}F + F\dot{F}(\mathcal{W}^2) + F\dot{F}(\mathcal{R}^N(g^N)^*(\nu\otimes\nu)) \tag{3–20}$$

where $\mathcal{L} = \dot{F}g^*\mathrm{Hess}_{\nabla}$.

Proof : The evolution equations for metric and normal follow as in section I. The evolution of II can be calculated from the definition (2–2):

$$\begin{aligned}
\frac{\partial}{\partial t}\mathit{II}(u,v) =& \nabla^N_{F\nu} g^N\left(\nabla^N_{T\varphi(u)}T\varphi(v),\nu\right)\\
=& g^N\left(\nabla^N_{F\nu}\nabla^N_{T\varphi(u)}T\varphi(v),\nu\right) + g^N\left(\nabla^N_{T\varphi(u)}T\varphi(v),-T\varphi(\nabla F)\right)\\
=& g^N\left(\nabla^N_{T\varphi(u)}\nabla^N_{F\nu}T\varphi(v),\nu\right) + FR^N(u,\nu,v,\nu) + g(\nabla_u v,-\nabla F)\\
=& g^N\left(\nabla^N_{T\varphi(u)}\nabla^N_{T\varphi(v)}(F\nu),\nu\right) + FR^N(u,\nu,v,\nu) - d_{\nabla_u v}F\\
=& d_u d_v F - Fg(\mathcal{W}(u),\mathcal{W}(v)) + FR^N(u,\nu,v,\nu) - d_{\nabla_u v}F\\
=& \mathrm{Hess}_{\nabla} F(u,v) - F\mathit{II}^2(u,v) + FR^N(u,\nu,v,\nu)
\end{aligned}$$

The remaining evolution equations follow exactly as in (I.3–7). ∎

Lemma 3–21.

(3–22)

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{W}(u) =&(\mathcal{L}\mathcal{W})(u) + g^*\ddot{F}(\nabla\mathcal{W},\nabla\mathcal{W})(u)\\ &+\dot{F}(\mathcal{W}^2)\mathcal{W}(u) + \mathcal{R}^N((g^N)^*(\nu\otimes\nu)(\dot{F})\mathcal{W}(u)\\ &+2\mathcal{R}^N(\dot{F}^\dagger\circ\mathcal{W})(u) - \mathcal{R}^N(\dot{F}^\dagger)(\mathcal{W}(u)) - \mathcal{W}(\mathcal{R}^N(\dot{F}^\dagger)(u))\\ &+\mathcal{S}(\dot{F}^\dagger)(u) - \alpha\dot{F}(\mathrm{Id})(\mathcal{W}^2(u) + R^N(\nu,u,\nu))\end{aligned}$$

where $\mathcal{S} : T^*M\otimes TM \to T^*M\otimes TM$ *is defined by the following equation:*

$$g((\mathcal{S}(u\otimes v))(w),z) = \nabla^N R(w,z,u,v,\nu) - \nabla^N R(u,v,w,z,\nu).$$

Proof: Apply Simons' Identity (2–7) to the equation (3–19). ∎

The following result allows us to deduce evolution equations for the graphical parametrisation of lemma (3–2) from those given above:

Lemma 3–22. *Suppose Q is a scalar quantity defined on $M^n\times[0,T)$ which evolves under* (1–2) *by the evolution equation*

$$\frac{\partial}{\partial t}Q(x,t) = \mathcal{L}Q(x,t) + Z(x,t)$$

for some $Z : M^n\times[0,T)\to R$, and let $\chi : \Sigma\times[0,t_0)\to M$ be the diffeomorphisms given by lemma (3–2). *Define $\bar{Q} : \Sigma^n\times[0,t_0)\to R$ by*

$$\bar{Q}(\xi,t) = Q(\chi_t(\xi),t).$$

The following evolution equation holds:

(3–23)

$$\begin{aligned}\frac{\partial}{\partial t}\bar{Q}(\xi,t) =&\bar{\mathcal{L}}\bar{Q}(\xi,t) + \bar{Z}(\xi,t)\\ &+\frac{\dot{F}g^*(\alpha\mathrm{Id} + I\!I^{(s)} + (\mathcal{W}^{(s)})^\dagger\nabla s\otimes\nabla s + \nabla s\otimes(\mathcal{W}^{(s)})^\dagger\nabla s)}{1+|\nabla s|^2_{g^{(s)}}}\end{aligned}$$

where $\bar{Z}(\xi, t) = Z(\chi_t(\xi), t)$ *and* $\bar{\mathcal{L}} = \dot{F} g^* \mathrm{Hess}_{\nabla^{(s)}}$.

Proof : This follows directly from the equations (2–18) and (2–19) which give expressions for the difference in the connections ∇ and $\nabla^{(s)}$, and from the equation (3–11) which determines the gradient term arising from the diffeomorphism χ of theorem (3–2). ∎

4. Preserving Convexity and Pinching

In this chapter it is proved that a solution to (1–2) remains strictly α-convex, where $\alpha = \sqrt{K_1}$, and also that the shifted principal curvatures $\lambda_i - \alpha$ remain pinched. The proof is very similar to the corresponding estimate (I.4–1), but slightly more complicated.

Theorem 4–1. *Let φ be a solution of (1–2) on the domain $M^n \times [0,T)$. Then the maximal time of existence T is finite, and there exist constants $C > 0$ and $\beta > \alpha$ depending on φ_0, K_1, and L such that the following estimates hold:*

$$\begin{aligned} &\lambda_i(x,t) - \alpha > C(\lambda_j(x,t) - \alpha) \\ &\lambda_i(x,t) \geq \beta \end{aligned} \tag{4–2}$$

for all i and j, and all (x,t) in $M^n \times [0,T)$.

Proof : The equation (3–20) will enable us to prove both that α-convexity is preserved and that the maximal time T is finite: Since $\lambda_i \geq \alpha$, we obtain at a point where F attains its infinum, using a frame $\{e_i\}$ which diagonalises $\mathcal{W}$:

$$\begin{aligned} \frac{\partial}{\partial t}F \geq &F\sum_i \frac{\partial f}{\partial \lambda_i}(\lambda_i^2 + \sigma^N(\nu \otimes e_i)) \\ \geq &F\sum_i \frac{\partial f}{\partial \lambda_i}(\alpha(\lambda_i - \alpha + \alpha) - \alpha^2) \\ \geq &\alpha F\sum_i \frac{\partial f}{\partial \lambda_i}(\lambda_i - \alpha) \\ \geq &\alpha F^2. \end{aligned}$$

F has an initial strictly positive lower bound. The maximum principle applied to the above equation shows that this is preserved in time. Since F has bounded gradient and is homogeneous, positive, and zero on the boundary of Γ_α, it is comparable to the smallest shifted eigenvalue $\lambda_{\min} - \alpha$, and strict α-convexity is preserved. The maximum principle also proves that the time of existence T is finite, since the above inequality forces $\inf_M F$ to become infinite in finite time.

As in section I, we consider quantities of the form $\frac{Q}{F}$, where $Q = q \circ \lambda$ and q is an appropriate convex, homogeneous degree one function of $(\lambda_1 - \alpha, \dots, \lambda_n - \alpha)$. Note that $\frac{Q}{F}$ approaches infinity on the boundary of the cone Γ_α, so it is sufficient to find an upper bound. First consider the evolution equation for Q, which is calculated by applying the derivative $\dot{Q}$ to equation (3–22):

(4–3)

$$\begin{aligned}\frac{\partial}{\partial t}Q =& \mathcal{L}Q + (\dot{Q}\ddot{F} - \dot{F}\ddot{Q})(\nabla\mathcal{W}, \nabla\mathcal{W}) + Q\dot{F}(\mathcal{W}^2 + \mathcal{R}^N(g^N)^*(\nu\otimes\nu))\\ &+ \mathcal{S}(\dot{F}, \dot{Q}) + 2\mathcal{R}^N(\dot{F}\circ\mathcal{W})(\dot{Q}) - 2\mathcal{R}^N(\dot{Q}\circ\mathcal{W})(\dot{F})\\ &+ \alpha\left(\dot{Q}(\mathrm{Id})\dot{F}(\mathcal{W}^2 + \mathcal{R}^N(g^N)^*(\nu\otimes\nu) - \dot{F}(\mathrm{Id})\dot{Q}(\mathcal{W}^2 + \mathcal{R}^N(g^N)^*(\nu\otimes\nu)\right)\end{aligned}$$

where $\mathcal{S}$ is given in lemma (3–21). From this it is easy to calculate the derivative of $\frac{Q}{F}$:

(4–4)

$$\begin{aligned}\frac{\partial}{\partial t}\frac{Q}{F} =& \mathcal{L}\frac{Q}{F} + \frac{\dot{Q}\ddot{F} - \dot{F}\ddot{Q}}{F}(\nabla\mathcal{W}, \nabla\mathcal{W}) + \frac{2}{F}\dot{F}g^*(\nabla F\otimes\nabla(\frac{Q}{F}))\\ &+ F^{-1}\mathcal{S}(\dot{F})(\dot{Q}) + \frac{2}{F}\mathcal{R}^N(\dot{F}\otimes\mathcal{W})(\dot{Q}) - \frac{2}{F}\mathcal{R}^N(\dot{Q}\otimes\mathcal{W})(\dot{F})\\ &+ \frac{\alpha}{F}\left(\dot{Q}(\mathrm{Id})\dot{F} - \dot{F}(\mathrm{Id})\dot{Q}\right)(\mathcal{W}^2 + \mathcal{R}^N(g^N)^*(\nu\otimes\nu)).\end{aligned}$$

The various terms appearing here are easily estimated: First, concavity of f implies concavity of F, and convexity of q implies convexity of Q, so we have

$$\frac{\dot{Q}\ddot{F} - \dot{F}\ddot{Q}}{F}(\nabla\mathcal{W}, \nabla\mathcal{W}) \le 0.$$

The next term contains a gradient of $\frac{Q}{F}$, and so can be ignored when applying the maximum principle. The global supremum bound on $\nabla^N R^N$ gives the following estimate:

$$|\mathcal{S}(\dot{F})(\dot{Q})| \leq CL \sup_{\Gamma_\alpha} |\dot{F} \otimes \dot{Q}|$$

where C is a constant depending only on n. Note that $\dot{F}$ and $\dot{Q}$ are bounded if we assume both f and q satisfy condition (7) of (3–1). The next terms can be estimated using the following simple calculation which is valid in a normal coordinate system at a point where $\mathcal{W}$ is diagonal:

(4–5)

$$2\mathcal{R}^N(\dot{F} \circ \mathcal{W})(\dot{Q}) - 2\mathcal{R}^N(\dot{Q} \circ \mathcal{W})(\dot{F}) = \sum_{i,j}(\frac{\partial f}{\partial \lambda_i}\frac{\partial q}{\partial \lambda_j} - \frac{\partial q}{\partial \lambda_i}\frac{\partial f}{\partial \lambda_j})(\lambda_i - \lambda_j)\sigma^N(e_i \wedge e_j)$$

where $e_1, \ldots, e_n$ are unit eigenvectors of $\mathcal{W}$. A similar calculation applies to the last terms:

$$\begin{aligned}(\dot{Q}(\mathrm{Id})\dot{F} - \dot{F}(\mathrm{Id})\dot{Q})(\mathcal{W}^2 + g^* R^N(.,\nu,.,\nu)) &= \frac{1}{2}\sum_{i,j}(\frac{\partial q}{\partial \lambda_i}\frac{\partial f}{\partial \lambda_j} - \frac{\partial f}{\partial \lambda_i}\frac{\partial q}{\partial \lambda_j})(\lambda_i^2 - \lambda_j^2) \\ &+ \sum_{i,j}(\frac{\partial q}{\partial \lambda_i}\frac{\partial f}{\partial \lambda_j} - \frac{\partial f}{\partial \lambda_i}\frac{\partial q}{\partial \lambda_j})\sigma^N(\nu \wedge e_j).\end{aligned}$$

The second term here can be estimated using K_2, K_1, and (7) of (3–1). The first combines with (4–5) to give

$$\sum_{i,j}\left(\frac{\partial q}{\partial \lambda_i}\frac{\partial f}{\partial \lambda_j} - \frac{\partial f}{\partial \lambda_i}\frac{\partial q}{\partial \lambda_j}\right)(\lambda_i - \lambda_j)(\sigma^N(e_i \wedge e_j) + \alpha\frac{\lambda_i + \lambda_j}{2})$$

The last factor here is positive by the assumption of α-convexity and the definition of α and K_1. The remaining factors are negative since f is concave—compare (2–21). The following estimate is obtained:

$$\text{(4–6)} \qquad \frac{\partial}{\partial t}\frac{Q}{F} \leq \mathcal{L}(\frac{Q}{F}) + \frac{2}{F}\dot{F}g^*(\nabla F \otimes \nabla(\frac{Q}{F})) + \frac{C}{\beta - \alpha}.$$

The parabolic maximum principle now gives $\sup \frac{Q}{F} \leq C(1+t)$ which is bounded since we know the interval of existence is finite.

Note that a suitable function Q can always be found—for example, the function $Q = |\mathcal{W} - \alpha \mathrm{Id}|$ satisfies all the required conditions. ∎

An immediate corollary is that the map $\dot{F}$ remains comparable to the identity map throughout the period of existence of the solution.

5. Local Estimates

In this chapter Hölder estimates are found for the curvature of the immersions φ. This is essentially an application of the general results described in [**K**], but some care is required to apply these in the absence of a lower bound on the injectivity radii of N. This is accomplished here by using the graphical coordinates ψ introduced in (2–13) and (3–2), which are nondegenerate but may not be diffeomorphic. Once the apparatus of (3–2) is in place, the application of [**K**] presents no difficulties.

The analysis is simplified by considering the scaling properties of equation (1–2), which follow directly from the homogeneity condition (4) of (3–1):

Lemma 5–1. *Suppose* $\varphi : M^n \times [0,T) \to (N^{n+1}, g^N)$ *is a solution to equation* (1–2) *with speed* $f(\lambda)$. *For any constant* $A > 0$, *define* $\varphi^{(A)} : M^n \times [0, A^2T) \to (N^{n+1}, A^2 g^N)$ *by* $\varphi_t^{(A)}(x) = \varphi_{A^{-2}t}(x)$. *Then* $\varphi^{(A)}$ *is a solution to* (1–2) *with speed function* $f^{(A)}(\lambda) = f\left(\lambda - \frac{A-1}{A}\alpha\right)$, *which satisfies the homogeneity condition* (4) *of* (3–1) *with* $\alpha^{(A)} = \frac{\alpha}{A}$.

The first problem is to consider appropriate graphical coordinates, and to estimate the time of existence and other properties of the solution given in lemma (3–2). The previous lemma assists us by allowing us to consider only solutions which are rescaled to satisfy a curvature bound:

Lemma 5–2. *Let* $\varphi : M^n \times [-1,1] \to N^{n+1}$ *be a solution of the equation* (1–2) *with* $\sup_{M^n} |\mathcal{W}(x,0)| = \sup_{[-1,0]\times M^n} |\mathcal{W}(x,t)| = 1$, *and suppose* N *is such that* $\max\{K_1, K_2, L\} \le 1$. *Choose* x_0 *in* M^n, *and let* $P = T_{x_0}\varphi_0(T_{x_0}M^n) \subset T_{\varphi_0(x_0)}N$.

Let ψ be the graphical coordinates over P. Then on a domain $B_\delta(0) \times [-\tau, \tau] \subset P \times \mathbb{R}$ there exists a smooth function s corresponding to φ by equation (2–15), *and we have*

$$\sup_{B_\delta \times [-\tau,\tau]} |s| \le \epsilon$$
$$\sup_{B_\delta \times [-\tau,\tau]} |Ds| \le 1 \tag{5–3}$$
$$\sup_{B_\delta \times [-\tau,\tau]} |\mathcal{W}| \le 2.$$

Here δ is a constant depending only on n and f.

Proof : At the initial time we can construct the required map χ and function s giving the graphical parametrisation of φ_0: Set $s(0) = 0$ and $\chi(0) = x_0$, and extend according to the following differential equations:

$$\begin{aligned} \nabla^{(s)} s(\xi) &= (T_\xi \psi^{(s)})^{-1} \left(\hat{\nu}^{(s)}(\xi) - \frac{\nu(\chi(\xi))}{g^N(\nu(\chi(\xi)), \hat{\nu}^{(s)}(\xi))} \right) \\ T_\xi \chi(u) &= (T_{\chi(\xi)} \varphi_0)^{-1} \left(T_\xi \psi^{(s)}(u) + \nabla_u s(\xi) \hat{\nu}^{(s)}(\xi) \right) \end{aligned} \tag{5–4}$$

where $\nabla_u s$ in the second equation is calculated by the first equation. These expressions can be used to solve for s and χ along radial curves from the origin in P. The solutions s and χ along such a curve can be extended within the region of definition of ψ as long as $|s| < \epsilon$ and $|\nabla s|_{g^{(s)}}$ remains bounded, since $g^N(\nu(\chi(\xi)), \hat{\nu}^{(s)}(\xi))^{-1} = \sqrt{1 + |\nabla s|^2_{g^{(s)}}}$. We can estimate these on a small region as follows: The expressions (2–17) and (2–19) can be combined to give an expression for $|\mathcal{W}|$, using the estimates (2–14). Since $|\mathcal{W}| \le 1$, this gives an estimate of the form

$$|D^2 s| \le C(1 + |Ds|^2)^{\frac{3}{2}} \tag{5–5}$$

for some constant C. Since $|\nabla s|(0) = 0$, this gives a bound on $|Ds|$ on a ball of radius r_0 which does not depend on x_0 or φ. Note that this also implies a bound on $|\nabla s|_{g^{(s)}}$, since $g^{(s)}$ is uniformly equivalent to the metric on P in the region considered. Without loss of generality, let us assume that we have taken r_0 sufficiently small to ensure that $|Ds| \leq \frac{1}{2}$. By taking r_0 smaller if necessary, this also ensures $|s| \leq \frac{\epsilon}{2}$.

The next difficulty is to show that this solution s exists for a fixed time interval on a suitable region of P, and to estimate $|Ds|$ throughout the time interval. Note that equations (3–11) in the proof of lemma (3–2) show that the solution can be extended in time as long as $|s| < \epsilon$ and $|Ds|$ is bounded. To control the curvature on a small interval, we can use equation (4–3) with $Q = |\mathcal{W}|$:

$$\begin{aligned}\frac{\partial}{\partial t}|\mathcal{W}| &\leq \mathcal{L}|\mathcal{W}| + |\mathcal{W}|\dot{F}(\mathcal{W}^2) + C\\ &\leq \mathcal{L}|\mathcal{W}| + C|\mathcal{W}|^3 + C\end{aligned}$$

where C depends only on f and the pinching bound of (4–2). Since $\sup_{t=0} |\mathcal{W}| = 1$, we can find a small time interval on which $|\mathcal{W}| \leq 2$. On this time interval we also have a bound $F \leq \frac{|\mathcal{W}|}{\sqrt{n}} \leq \frac{2}{\sqrt{n}}$. On an interval $[0, \tau]$, the solution stays in a neighbourhood of width $\frac{2\tau}{\sqrt{n}}$ of the initial immersion φ_0. For τ sufficiently small, and considering only the smaller region of radius $\frac{r_0}{2}$ in P, this neighbourhood is contained in a strip about the intial function s_0, given by $s_0 - C\tau \leq s \leq s_0 + C\tau$ for some constant C, using the bound on $|Ds|$ at the initial time. Clearly we can choose τ small enough to ensure that $|s| < \epsilon$ on this region. Now we use the bound (5–5) again, in the form

$$|D|Ds|| \leq C(1 + |Ds|^2)^{\frac{3}{2}}.$$

Integrate along a curve γ which begins at some point in $B_{\frac{r_0}{2}}(0) \subset P$, and follows the direction of steepest ascent of s. First we have an estimate on $|Ds|$ from below for small distances:

$$|Ds|(r) \geq \frac{A - Cr}{\sqrt{1 - (A - Cr)^2}}$$

where $A = \frac{|Ds|(0)}{\sqrt{1+|Ds|^2}}$; this holds for $A - Cr \geq 0$. Integrating again we obtain the estimate

$$s(\gamma(r)) - s(\gamma(0)) \geq C^{-1}\left(\sqrt{1-(A-Cr)^2} - \sqrt{1-A^2}\right).$$

Suppose $|Ds|(\gamma(0)) > 1$. Then for a distance r no greater than $C^{-1}(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}})$ we have the estimate $s(\gamma(r)) - s(\gamma(0)) \geq \frac{r}{\sqrt{2}}$. However the estimates obtained above ensure that $s(\gamma(r)) - s(\gamma(0)) \leq C\tau + \frac{r}{2}$, using the gradient bound at the initial time. Consider points which are contained in the ball of radius $\frac{1}{3}r_0$, and paths γ of fixed length r no greater than the minimum of $C^{-1}(\frac{1}{2} - \frac{1}{3})$ and $\frac{1}{6}r_0$. The endpoint of any such curve is still contained in the ball of radius $\frac{1}{2}r_0$, but has $s(\gamma(r)) - s(\gamma(0)) \geq \frac{1}{\sqrt{2}} > \frac{1}{2}r + C\tau$ provided we restrict to a time interval of length no greater than $C^{-1}r\frac{\sqrt{2}-1}{2}$.

The same techniques show that the solution can be extended backward in time to $-\tau$, since we have assumed a curvature bound on $[-\tau, 0]$. ∎

Now we are in a position to begin applying estimates from [**K**]. Note that we have existence of s on a region which is independent of any bound on the injectivity radii. The first estimate we obtain is a bound on the oscillation of the curvature:

Lemma 5–6. *Under the conditions in lemma (5–2), there exists a positive function $\sigma : (0,1] \to \mathbb{R}$ such that*

$$\inf_{(\xi,t)\in B_{\delta/2}\times[\gamma\tau,\tau]} F(\xi,t) \geq \sigma(\gamma)F(0,0) \tag{5–7}$$

for all $\gamma \in (0,1]$.

Proof : The previous lemma allows us to apply directly the following Harnack inequality due to Krylov and Safonov ([**KS**]; see also [**K**], section (3.1)):

Lemma 5–8. *Let u be a positive solution in $W^{1,2}(B_1(0)\times[-1,1]$ to the equation*

$$\frac{\partial}{\partial t}u = a^{ij}(x,t)D_iD_ju + b^i(x,t)D_iu + c(x,t)u \tag{5–9}$$

on the domain $B_1(0)\times[-1,1]\subset\mathbb{R}^n\times\mathbb{R}$, where the coefficients are measureable, bounded, and uniformly elliptic:

$$\begin{aligned} &\underline{C}|v|^2 \le a^{ij}v_iv_j \le \overline{C}|v|^2 \\ &|b| \le C \\ &|c| \le C \end{aligned} \tag{5–10}$$

for all (x,t) in $B_1(0)\times[-1,1]$ and $v\in\mathbb{R}^n$. Then there exists a constant K depending only on n, $\underline{C}$, $\overline{C}$ and C such that

$$\inf_{B_{\frac{1}{2}}(0)} u(x,1) \ge \frac{u(0,0)}{K}. \tag{5–11}$$

If $\underline{C}$, $\overline{C}$, and C change within a bounded range, then so does K.

An application this lemma followed by rescaling of either space or time variables gives a more general result. In view of the estimates (2–14) which control the map ψ, and the bounds on height, gradient and curvature (5–3), this lemma can be used immediately to obtain the desired result. ∎

Lemma 5–12. *Under the conditions of lemma (5–2) the following estimate holds if x_0 is chosen so that $\sup_{M^n}|\mathcal{W}(x,0)| = |\mathcal{W}(x_0,0)| = 1$:*

$$\inf_{(y,t)\in B_{\delta/2}(x)\times[\tau/2,\tau]} F(y,t) \ge C \tag{5–13}$$

where C is a function of n, f, and $d_0(x,x_0)$ (Here d_0 is the distance in M^n with respect to g at time 0).

Proof : This result follows by repeated application of the previous lemma—for points near x_0, a single application suffices. For points further away, several applications on shorter time intervals give the result. ▮

Lemma 5–14. *Under the conditions of lemma* (5–12),

$$\|\mathcal{W}\|_{C^{0,\beta(x)}(B_\delta(x)\times[\frac{3\tau}{4},\tau]} \leq C(x). \tag{5–15}$$

where $\beta(x)$ and $C(x)$ are functions of n, f, and $d_0(x, x_0)$.

Proof : For this result we can apply a more sophisticated result from [**K**] (section (5.5)) which gives Hölder estimates for the second derivatives of solutions to uniformly parabolic equations

$$\frac{\partial}{\partial t}u = \mathcal{F}(D^2u, Du, u, x, t)$$

where F is convex (or concave) in the second derivatives, provided some other conditions are satisfied involving boundedness of the derivatives of $\mathcal{F}$ with respect to other arguments. Our previous lemma ensures that the curvature is bounded above and below on each region we consider. This guarantees that all the required conditions are satisfied, and the result follows. ▮

6. Convergence

In this chapter we apply the estimates from the previous chapter to complete the proof of theorem (1–5). This involves showing convergence to a sphere on a subsequence of times under an appropriate rescaling (which uses a recent result of Hamilton [**Ha6**]), and then deducing the convergence for other times (which is in most respects analogous to the proof in the Euclidean case). Before we can carry out this program, we require the following result, which guarantees existence of the solution as long as the curvature remains bounded:

Theorem 6–1. *Suppose $\varphi : M^n \times [0, t_0) \to N^{n+1}$ is a smooth α-convex solution to (1–2), and $\sup_{M^n\times[0,t_0)} F < \infty$. Then φ extends uniquely to $M^n \times [0, t_1)$ for some $t_1 > t_0$.*

Proof: The result (5–12) ensures that we have C^α estimates for the curvature of φ on the domain $M^n \times [0, t_0)$. Note that the distance moved by any point is bounded by $t_0 \sup_{M^n\times[0,t_0)} F < \infty$, so the image of φ is contained in a compact set of N on this time interval. Consequently we have bounds on all the higher derivatives of the Riemann tensor of N. Standard Schauder estimates therefore provide bounds on all the derivatives of the curvature of φ. This ensures C^∞ convergence to an immersion φ_{t_0} (see for example [**Hu1**], section 8). The short time existence result (3–14) now applies to extend the solution to a longer time interval. ∎

The estimates of chapter 5 are enough to prove convergence in a restricted sense: We consider a subsequence of times $\{t_k\}$ approaching the maximal time of existence T of φ, chosen such that the following holds for a corresponding sequence of points x_k in N:

$$\sup_{M^n \times [0,t_k]} |\mathcal{W}|(x,t) = |\mathcal{W}|(x_k, t_k). \tag{6–2}$$

The existence of such a sequence is guaranteed by the result (6–1).

For each k we rescale the metric g^N on a time interval about t_k to make φ satisfy the curvature bound required for the application of lemma (5–1). Then we use lemma (5–1) with $A = A_k = \sup_{[0,t_k]} |\mathcal{W}|$, and proceed with the estimates of chapter 5, obtaining Hölder estimates on the curvature on a time interval of rescaled duration τ, depending only on the rescaled distance from the point x_k.

For each k, we choose an isometry from $\mathbb{R}^{n+1}$ to $T_{\varphi(x_k)}N$. In this way we identify the tangent spaces to N at each of these points. Note that the exponential map at $\varphi(x_k)$ is nondegenerate on a ball of (rescaled) radius $r_0 A_k$ for some fixed $r_0 > 0$ depending on K_1 and K_2. Since A_k is unbounded as k becomes large, the exponential map is eventually nondegenerate on arbitrarily large regions of $\mathbb{R}^{n+1}$ under this identification. Furthermore, the curvature bounds (1–1) show that the metric induced on $\mathbb{R}^{n+1}$ by the exponential maps converges in C^3 to the flat metric as k tends to infinity. Although the exponential map may not be diffeomorphic on these regions, we can use the nondegeneracy to obtain a family of hypersurface in $B_{r_0 A_k}(0) \subset \mathbb{R}^{n+1}$ which corresponds to the family $\varphi(M)$ under the exponential map. This is given by the solution to the following differential equation for immersions $\tilde{\varphi}$ into $\mathbb{R}^{n+1}$:

$$T\tilde{\varphi} = \left(T_{\tilde{\varphi}} \exp_{x_k}\right)^{-1} \circ T\varphi. \tag{6–3}$$

$$\tilde{\varphi}(x_k, t_k) = 0$$

The estimates from lemma (5–14) give $C^{2+\beta}$ estimates on each ball B_r in $\mathbb{R}^{n+1}$, independently of k. Hence for each positive integer R we can find a subsequence

$\{t_{k_R}\}$ of $\{t_k\}$ for which the families of hypersurfaces converge to a $C^{2+\beta}$ family of hypersurface of R^{n+1}. Furthermore we can arrange that $\{t_{k_{R+1}}\}$ is a subsequence of $\{t_{k_R}\}$ for each R. Taking a diagonal subsequence $\{t_{k_k}\}$, we obtain convergence to a limiting family of complete hypersurface in $\mathbb{R}^{n+1}$. Each hypersurface in this family satisfies the estimates (5–15), depending only on the distance from the origin. Furthermore, the limiting family consists of strictly convex hypersurfaces with curvature bounded below by the estimate (5–13), depending only on distance from the origin. The curvature of the hypersurfaces is also bounded ($|\mathcal{W}| \leq 2$), and the family is a solution to the equation (1–2) with $\alpha = 0$. It follows (again using the estimates of chapter 5 and Schauder theory) that the limit hypersurfaces are smooth.

We can now employ the following recent result due to Hamilton [**Ha6**]:

Theorem 6–4. *A complete, smooth, strictly convex hypersurface with pinched principal curvatures in Euclidean space is compact.*

It follows immediately that the solutions φ are boundaries of small immersed balls in N for sufficiently large times. In particular, the solution remains in a compact subset of N for the length of its existence. This implies that the solution converges for a subsequence of times to some point of N, since the hypersurfaces approach a compact hypersurface after arbitrarily large rescaling, and so have diameter tending to zero. It follows that we have convergence to a point of the whole solution, since later hypersurfaces are contained by earlier hypersurfaces.

Note that this result immediately gives us uniform estimates in C^∞ for the rescaled hypersurfaces, since the solution remains in a compact subset of N, and we have uniform estimates on all the derivatives of the Riemann tensor of N on this region. This also implies that we have convergence to the limiting hypersurfaces in C^∞ on the subsequence of times.

The limit hypersurfaces must in fact be spheres. This follows from the evolution equation (4–4) for the pinching quotient $\frac{Q}{F}$: In the limit, the maximum of this quotient is nonincreasing. By the strong maximum principle, the maximum is strictly decreasing unless $\frac{Q}{F}$ is constant. But if the maximum decreases on the family of limiting hypersurfaces, we have a contradiction to the convergence (note that the quantity $\frac{Q}{F}$ is unaffected by the rescaling process). Hence $\frac{Q}{F}$ is constant in the limit, for any Q satisfying the conditions of chapter 4. But then in equation (4–4), the negative second term must also vanish, which implies that the limiting hypersurfaces have constant curvature and are therefore spheres.

The Harnack estimate (5–13) gives bounds below on the rescaled curvature at each of the times t_k, since the diameter of the hypersurface is finite. Since the (unrescaled) minumum of the curvature is nondecreasing by the maximum principle applied to equation (3–20), this ensures that after some sufficiently large time, the hypersurfaces are strictly convex and pinched with respect to the flat metric on $\mathbb{R}^{n+1}$. The proof now proceeds exactly as in section I, chapter 7.

7. Extensions and Applications

In this chapter we conclude with some extensions to slightly different flows, and some applications to geometry.

Theorem 7–1. *For any strictly α-convex initial immersion φ_0, there exists a unique smooth solution φ on a finite time interval $[0, T)$ to the equation (1–2) with speed f satisfying conditions (3–1) with (4) replaced by homogeneity of degree one in λ. The immersions φ_t converge to a point of N and become spherical as in theorem (1–5).*

Proof: Equation (3–20) still ensures that convexity is preserved (although α-convexity need not be preserved), with a bound below on the principal curvatures decaying exponentially in time. The theorem (6–1) still holds, showing that a solution which has bounded curvature on a finite time interval can be extended further. On any finite time interval equation (4–4) still yields a pinching estimate. Therefore it is sufficient to show that the interval of existence of the solution is finite—the proof then proceeds exactly as before.

First note that φ_0 encloses an immersed disc, by theorem (1–5); we can consider the evolution as taking place on the disc itself, in which φ_0 is embedded. The solution φ to this equation immediately becomes enclosed by the solution $\varphi^{(\alpha)}$ of the nonhomogeneous equation. The solutions also remain disjoint: Suppose the two solutions touched again. At the point where this occurs the curvature of the outer hypersurface $\varphi^{(\alpha)}$ is no greater than the curvature of the inner hypersurface φ. Hence the rate of change of the distance between the hypersurfaces at such a point can be estimated as follows:

$$\begin{aligned}
\frac{\partial}{\partial t} d(\varphi, \varphi^{(\alpha)}) &\geq f(\mathcal{W}) - f^{(\alpha)}(\mathcal{W}^{(\alpha)}) \\
&\geq f(\mathcal{W}^{(\alpha)}) - f(\mathcal{W}^{(\alpha)} - \alpha \mathrm{Id}) \\
&\geq \alpha \inf_{\Gamma_+} \dot{F}(\mathrm{Id}) \\
&\geq \alpha
\end{aligned}$$

where we have used the inequality (2–20) in the last step. This is a contradiction since $\frac{\partial}{\partial t} d \leq 0$ at a newly attained minimum of d. Since $\varphi^{(\alpha)}$ contracts to a point in finite time, φ can only exist for a finite time. ∎

Note that this proof depends very strongly upon the result for the nonhomogeneous equation proved in this section. I know of no way to prove this result directly.

The first application I wish to discuss is a simple proof of the 1/4- pinching sphere theorem of Klingenberg, Berger and Rauch. This proof uses a method devised by Gromov and employed by Eschenburg [**Es**].

Theorem 7–2. *Let N be a compact simply connected smooth Riemannian manifold with sectional curvatures in the range $\frac{1}{4} < \sigma^N \leq 1$. Then N is diffeomorphic to a twisted sphere.*

Proof : Choose a point x_0 in N, and consider exponential spheres about x_0. We consider these as immersed spheres given by immersions φ_s where s is the distance parameter. These immersions are related by the equations

$$\frac{\partial}{\partial s} \varphi = \nu.$$

The change in the curvature and the metric on these spheres in given by the following equations, the proof of which is identical to the proof of theorem (3–15):

$$\frac{\partial}{\partial s} g(u,v) = 2I\!I(u,v) \tag{7–3}$$

$$\frac{\partial}{\partial s} \mathcal{W}(u) = -\mathcal{W}^2(u) - R^N(\nu,u,\nu) \tag{7–4}$$

Using the assumptions on the curvature of N, we obtain the following estimates for the maximum and minumum principal curvatures of the exponential spheres:

$$\lambda_{\max} < \frac{1}{2}\cot(\frac{1}{2}s) \tag{7–5}$$

$$\lambda_{\min} \geq \cot(s) \tag{7–6}$$

It follows that the exponential spheres are nondegenerate for any $s < \pi$: The equation (7–3) for the metric gives a bound on the metric as long as $|\mathcal{W}|$ remains finite for expanding exponential spheres. The strict inequality in (7–5) implies that there is some distance $s < \pi$ for which $0 > \lambda_{\max} \geq \lambda_{min} > -\infty$, and hence the exponential sphere at this distance is strictly convex in the outward direction. It follows from theorem (1–5) that this sphere bounds a disc in N. This gives an expression for N as a union of two discs by a diffeomorphism from one boundary to the other. ∎

The result from theorem (1–5) in the general case allows negative curvature in N. We can use this to prove the following "dented sphere theorem" which generalises the $\frac{1}{4}$-pinching theorem above:

Theorem 7–7. *Let N be a compact smooth simply connected Riemannian manifold with sectional curvatures bounded below by some constant $-\alpha^2$. Let $\epsilon \in (\frac{1}{2}, 1)$ be such that $\epsilon\cot(\epsilon\pi) < -\alpha$, and let $\rho \in [\frac{\pi}{2}, \pi)$ be such that $\epsilon\cot(\epsilon\rho) = -\alpha$. If there is a point x_0 in N such that $\epsilon < \sigma^N \leq 1$ on the ball $B_\rho(x_0)$, then N is diffeomorphic to a twisted sphere.*

Note that for any bound below for the sectional curvatures of N, one can find a pinching ratio ϵ and a radius ρ which satisfy the conditions here. If α becomes very large, then ϵ must be taken very close to 1 and ρ must be taken very close to π.

Proof: This is exactly analogous to the previous theorem. If we take expanding exponential spheres about the point x_0, the evolution of minimum and maximum principal curvatures can be estimated by the following equations:

$$\frac{\partial}{\partial s}\lambda_{\max} < -\lambda_{\max}^2 - \epsilon^2 \tag{7–8}$$

$$\frac{\partial}{\partial s}\lambda_{\min} \geq -\lambda_{\min}^2 - 1 \tag{7–9}$$

This gives the following estimates for balls of radius less than or equal to ρ:

$$\epsilon\cot(\epsilon s) > \lambda_{\max} \geq \lambda_{\min} \geq \cot(s). \tag{7–10}$$

At distance ρ the hypersurface is still nondegenerate, and is α-convex in the outward direction, but possibly not strictly α-convex. However, since N is smooth, there is some short distance beyond ρ on which the sectional curvatures are positive. Hence by taking a distance s slightly larger than ρ, we obtain a nondegenerate, strictly outward α-convex hypersurface. By theorem (1–5), this is the boundary of a disc, and the result follows. ∎

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